P. D. E. in Planar Regions with Fractal Boundary

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To Caroline Series, without whose constant support and valuable criticism this thesis would not have been written.

Introduction

The classical theory of P.D.E. in a bounded domain $\Omega \subset \mathbf{R}^2$ has always presupposed some degree of smoothness for the boundary of Ω : the main theorems include it in their hypotheses.

Hard analysis uses smoothness of the boundary of Ω at many stages in the solution of a P.D.E. problem posed on Ω . On the contrary, soft analysis--perhaps less precise, but sometimes more general--uses the smoothness of the boundary much less: it takes general properties of P.D.O.s and it works in an abstract space of operators. ...But, when focusing on some of these properties--the theorem of Rellich, the theorem of extension of Sobolev...--that smoothness of the boundary promptly appears as an inevitable part of the hypotheses.

The theorem of Rellich is one of the tools most used: in standard texts on elliptic P.D.E.s --such as that of Agmon--it is quoted at the very beginning; it is used at every stage of the theory of perturbation; it was used in order to solve classical problems like the distribution of eigenvalues of an elliptic P.D.O. [1], or the problem of the asymptotic behaviour of the scattering phase for exterior domains posed by Majda [2] and Kato [3]; ... it is in the frontier between P.D.E. and functional analysis; and classical texts on functional analysis (e.g. [4]) dedicate whole chapters to it.

There are several degrees of smothness (see below) of the boundary of Ω : a closed bounded curve $\partial\Omega$: C^{∞} , C^1-extended (e.g. so as to include a square), class C , cone property, segment property; each more relaxed than the former.

Subsequent proofs of the <u>same</u> theorem in P.D.E. relax more and more the conditions of smoothness used, making the theorem more general.

However, all the conditions quoted above imply that $\partial\Omega$ is of dimension 1.

We purpose to introduce the subject of P.D.E. in a planar region Ω whose boundary $\partial\Omega$ is <u>not</u> of dimension 1, but an object of dimension $d \in (1, 2)$.

The objects of dimension $d \in (1, 2)$ we will be working with, will be constructed by iteration *ad-infinitum* of a <u>process of replacement</u> that is described thus:

a segment is divided into n equal segments, and is replaced by a polygonal arc having the same end points as the given segment; this polygonal arc has $N = n^d$ segments of the same length as the n segments into which the original segment was divided.

The process of replacement starting from any segment, and its iteration *ad-infinitum* give rise to an object of Hausdorff dimension

$d = \log N/\log n$, as described in [5].

A well known example is the Koch snowflake--see figure (1)--: the process of replacement is iterated *ad-infinitum* simultaneously in all segments of an initial triangle, yielding a closed curve.

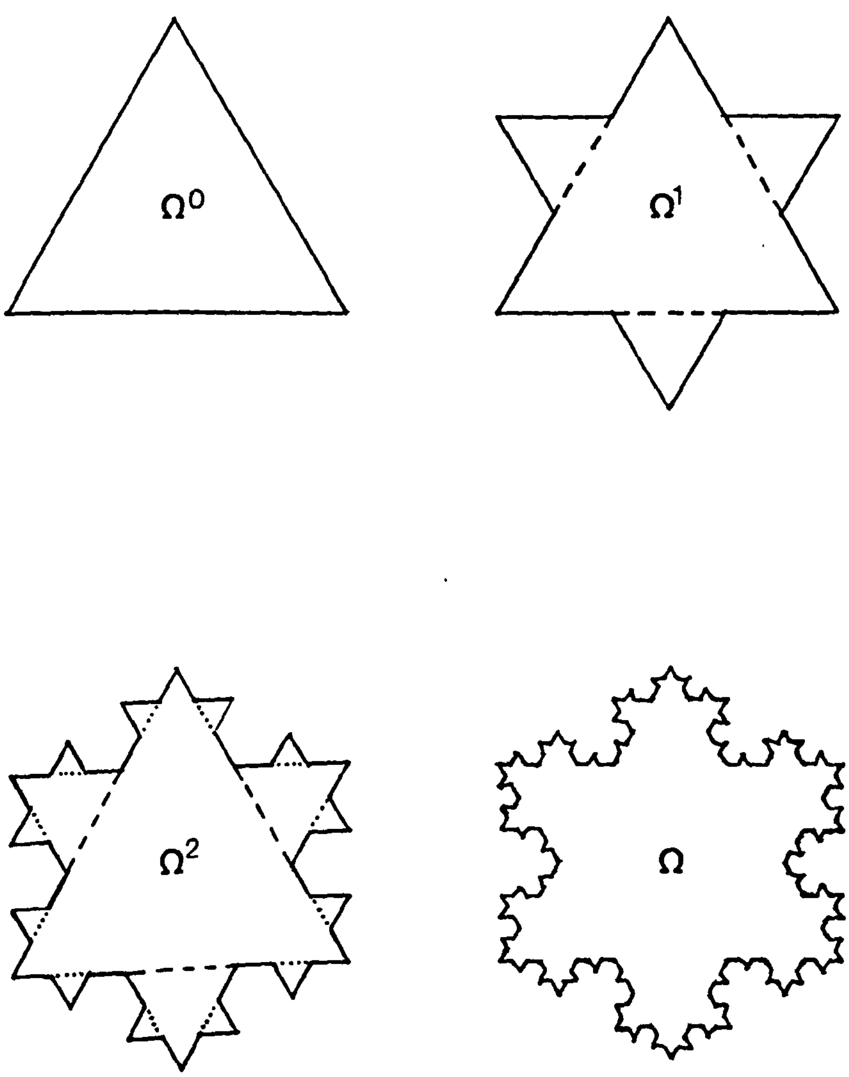


Figure 1

In general, we will consider a fractal curve $\partial\Omega$ as constructed in stages from an initial polygon, by successive iterations of the process of replacement carried out simultaneously in all segments of that polygon.

However, since we will be doing analysis in the open region Ω whose boundary is the fractal curve $\partial\Omega$, we are more interested in considering the region Ω itself as built in stages, as the union of regions Ω^i, $i \in \mathbb{N}$,

$$\Omega^0 \subset \Omega^1 \subset \Omega^2 \ldots \subset \Omega^p \subset \ldots \quad ;$$

$\partial\Omega^p$ is a polygon, such that the end points of any connected component of $\partial\Omega^{p+1} \cap \partial\Omega^p$ are points in $\partial\Omega$.

For the Koch snowflake, Ω^0, Ω^1, and Ω^2 are seen in figure (1).

We must restrict the process of replacement to those constructions which yield a closed curve $\partial\Omega$ such that Ω is a connected open region: the theorem of Rellich, among others in P.D.E., is not valid without this restriction.

For similar reasons we must restrict ourselves to bounded regions.

We will prove the theorem of Rellich for regions Ω, $\partial\Omega$ a fractal of dimension $d \in (1,2)$, by estimating the index $\Gamma^\varepsilon(\Omega)$ (see below), and exhibiting its (smooth) dependence on the dimension d.

The proof will meander through the resolution of some little problems in P.D.E. in such regions Ω with fractal boundary, giving a short list of techniques to work in analysis, in the interior of such a region Ω.

In the literature there are numerous counterexamples to the theorem of Rellich: they are based on curves $\partial\Omega$ which are not smooth at just one point; so that fractal curves would seem naturally excluded ...; chapters 0 and 1 deal with the preliminaries and with those counterexamples. Chapter 2 still focusses on smoothness: a fractal curve is C^1-extended or of class C at no point, ... but it does have points which are vertices of cones contained in Ω, or end points of segments contained in Ω. We prove that the Haussdorf d-measure of the sets of all these "smooth" points is zero. Chapter 3 is entirely of a technical nature: it reduces the problem to simpler regions, simpler functions and other such simplifications. Chapter 4 proves the theorem of Rellich for a particular class of functions. Chapter 5 describes some types of foliations of Ω; the leaves of the foliations have starting points in Int Ω, and end points in $\partial\Omega$: these foliations are our main tool for doing analysis in Int Ω, $\partial\Omega$ being a fractal. Chapter 6 gives a connection between the size (in a sense defined below) of a level set of a function defined on Ω ($\partial\Omega$ either smooth or a fractal) and the gradient of that function when restricted to its values under that level. Chapter 7 proves that, given a function u supported on Ω with fractal boundary, there exists another function $\bar{u}$, locally linear (in a sense defined below), with roughly the same support (see below), having bigger L^2 norm than u, but smaller gradient. In the same chapter we finish our proof of the theorem of Rellich. Finally, chapter 8 deals with the pathological case in which the dimension $d = 2$, and proves some pertinent points about Osgood monsters.

Chapter 0

Some Notation :

Throughout , Ω , Ω^p, $p \in \mathbf{N}$, will always denote bounded connected open regions in $\mathbf{R}^2$.

When we write $\partial\Omega$ smooth we mean : the curve $\partial\Omega$ is C^1 extended (so as to include sharp corners, as in the case of a square), or is of class C , or has the cone property , or has the segment property , in the sense specified in the following definitions :

$\partial\Omega$ of class C : Given any $p \in \partial\Omega$, there exists an open neighbourhood U(p) of p , a cartesian coordinate system (x, y) and a function f_p in that system, such that

$$\partial\Omega \cap U_p = \{ (x, f_p(x)) : x \in s_p \}$$

where s_p is an open segment, and f_p is a real valued continuous function on the clausure of s_p.

Ω has the cone property if there exists a finite cone c such that each point $x \in \Omega$ is the vertex of a finite cone c_x contained in Ω and congruent to c (c_x not necessarily obtained from c by parallel translation, just by rigid motion).

Ω has the segment property if for every $x \in \partial\Omega$, there exists an open set U_x and a non zero vector y_x such that $x \in U_x$, and if $z \in (\Omega \cup \partial\Omega) \cap U_x$, then $z+ty_x \in \Omega$, $t \in [0,1]$.

Let $\{A_i\}_{i\in\mathbb{N}}$, $A_i \subset \mathbb{R}^2$ for $i \in \mathbb{N}$.
We denote the disjoint union of the sets A_i , $i\in\mathbb{N}$, by

$$\sum_{i\in\mathbb{N}} A_i \ .$$

We write

$A = B + C$; A, B, C, sets in $\mathbb{R}^2$, and we say that B and C are non rampant , if

a) $\mathrm{Int.}B \neq \emptyset$, $\mathrm{Int.}C \neq \emptyset$

b) $\mathrm{Int.}B \cap \mathrm{Int.}C = \emptyset$

c) $\mu^2(\overline{B} \cap \overline{C}) = 0$, where μ^2 is the Lebesgue measure in $\mathbb{R}^2$; $\overline{B} = \mathrm{cl.}B$; $\overline{C} = \mathrm{cl.}C$.

When we write $A = B + C$; $A \subset \mathbb{R}^2$, $B \subset \mathbb{R}^2$, $C \subset \mathbb{R}^2$, it will be irrelevant for our purposes if they are open , closed , or neither.

For example , the Koch snowflake can be regarded as the non-rampant union of triangles of decreasing size and similar shape.

Next we will write any Ω , $\partial\Omega$ a fractal curve , in a number of ways as a non rampant sum of subsets , just as the Koch snowflake as a sum of triangles.

In what follows , we intend to give a terminology that answers the question : "where are we , inside the region Ω ? " : we want to be able to refer to different regions inside Ω , e.g. the different triangles added to Ω^p in order to build Ω^{p+1} in the Koch snowflake.

For any set $A \subset \mathbb{R}^2$, cc(A) will denote a connected component of A .

A tail T^1 of size 1 , is any

$$cc\,(\,\Omega - cl.\Omega^0\,) \quad \text{of maximal size;}$$

similarly, a tail T^p of size p , $p \in \mathbb{N}$, is any

$$cc\,(\,\Omega - cl\,\Omega^{p-1}\,) \quad \text{of maximal size.}$$

Notice that the tails decrease in size as the index $p \in \mathbb{N}$ increases.

For the Koch snowflake, with the Ω^0 shown in figure (1) we can write Ω as a non rampant sum of Ω^0 and tails of every size.

Remark :

Notice that, if we reduce the size of Ω^0 to a smaller one $\Omega^{0'} \subset \Omega^0$ for the Koch snowflake, as shown in figure (2), then we can write Ω as a non rampant sum of $\Omega^{0'}$ and six tails T^1 of size 1 only.

In the general case we can always choose

$\Omega^{0'} \subset \Omega^0$, such that

$$\Omega = \Omega^{0'} + \sum_i T_i^1 \quad ,$$

where the T_i^1 are tails of size 1 and the sum is finite.

The classical theorem of Rellich states:

If $\partial\Omega$ is sufficiently smooth, then the inclusion

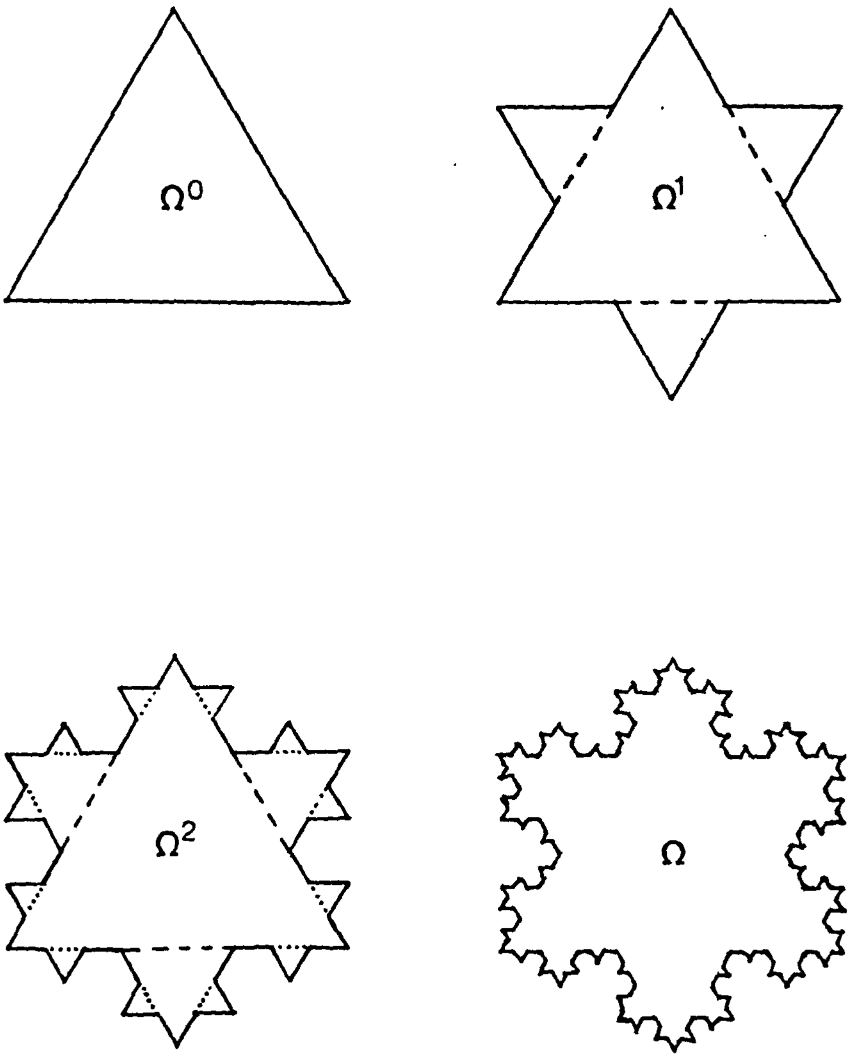


Figure 1

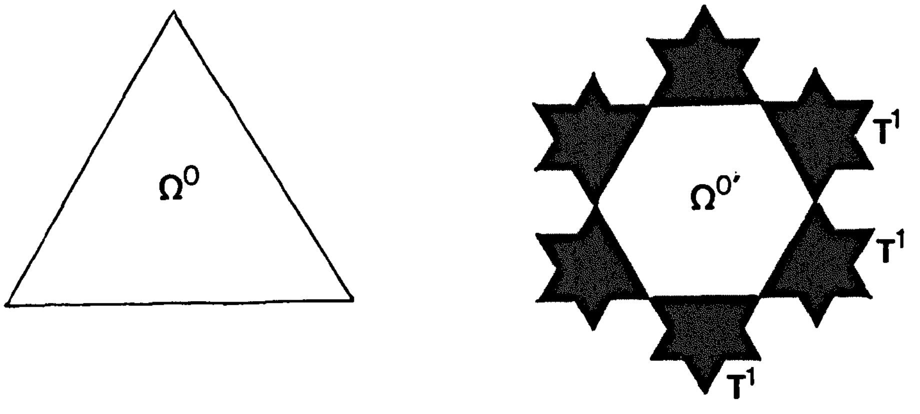


Figure 2

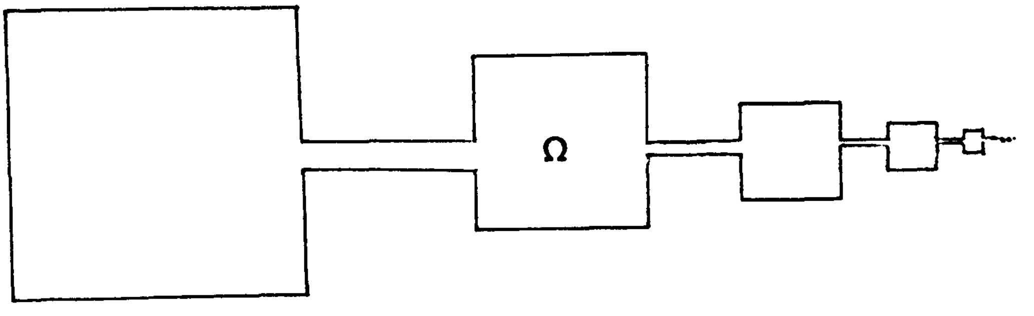


Figure 3

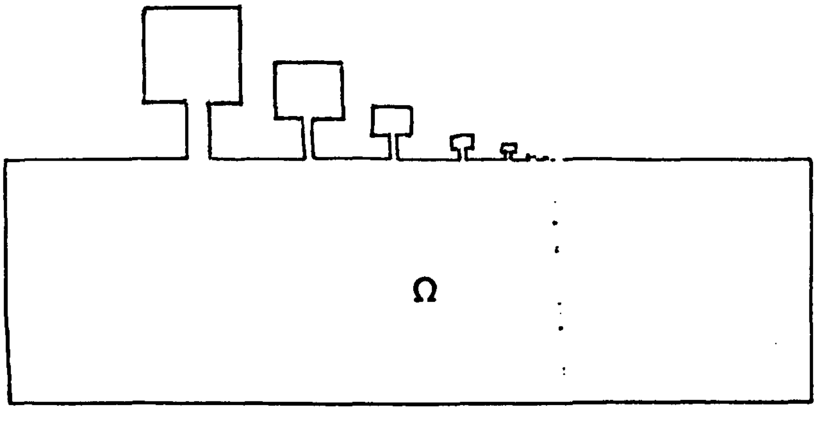


Figure 4

$$L_1^2(\Omega) \hookrightarrow L^2(\Omega)$$

is a compact operator.

A word about the Sobolev spaces L_k^p will be said below.

We will extend the theorem of Rellich to regions Ω , $\partial\Omega$ a fractal curve.

The decomposition into an initial body and tails of the same or different sizes stated above has the following motivation:

The work of Amick [6] proves that the part of Ω that determines the validity of the theorem of Rellich, is the subregion very near the boundary $\partial\Omega$. In that sense, $\Omega^{0'}$ or Ω^0 will be the safe subregion of Ω , and the tails of decreasing size will be the dangerous subregions of Ω near the boundary $\partial\Omega$.

In fact, the first part of the work will be: to reduce the study of a general function, supported in <u>all</u> Ω , to the study of functions supported in different tails.

The Koch snowflake has all tails with the same shape, irrespective of their size. We refer to this property as <u>self similarity of shape</u>, irrespective of scale.

Notice that, in the general case, we have a finite number of self similar possible shapes of tails. This finite number depends on the particular fractal $\partial\Omega$.

In what follows, we list the types of counterexamples to the Rellich theorem common in the literature; that is, the types of regions Ω for which the Rellich theorem is not true, and show that the self similarity of shape irrespective of scale, just referred to, sharply separates our regions Ω , $\partial\Omega$ a fractal, from all those counterexamples.

Let us recall that the (classical) theorem of Rellich establishes a sufficient condition (the smoothness of $\partial\Omega$) for the compactness of the inclusion

$$L_1^2(\Omega) \hookrightarrow L^2(\Omega) \ .$$

Amick [6] has proved an equivalent condition for that compactness:

Theorem : Let Ω be a bounded region in $\mathbf{R}^2$. Then

$$L_1^2(\Omega) \hookrightarrow L^2(\Omega)$$

is compact if and only if

$$\Gamma^{\varepsilon}(\Omega) \to 0 \ , \text{when} \ \varepsilon \to 0 \ ;$$

where $\Gamma^{\varepsilon}(\Omega) =$

$$\sup_{\substack{u \in L_1^2(\Omega) \\ u \neq 0}} \frac{\displaystyle\iint_{\Omega^{\varepsilon}} u^2}{\displaystyle\iint_{\Omega} u^2 + \iint_{\Omega} \operatorname{grad}^2 u}$$

$$\text{and} \ \ \Omega^{\varepsilon} = \{ x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) < \varepsilon \} .$$

The apparent necessity of the condition of smoothness on $\partial\Omega$ is seen by Edmunds with a counterexample introduced by Amick [6], that has become known as "rooms and passages" in the literature.

We now sketch how Amick's counterexample works:

As shown in figure (3), the region Ω is a non rampant union of square rooms and narrow passages joining the rooms. The length of the side of room R_i equals the length of the passage P_{i-1}. The rooms (and passages) decrease in size, and are infinite in number.

If the length of the passage P_i is η_i, then the width of the passage P_i is $(\eta_{i+1})^{3+\alpha}$, $\alpha \geq 0$; where

$$\lim_{i \to \infty} \eta_i = 0$$

For $\varepsilon > 0$ arbitrary, let $i \in \mathbb{N}$ be such that $\eta_i < \varepsilon$, and choose

$$u_i \in L_1^2(\Omega)$$

such that

$$u_i \equiv 1 \text{ in } R_i$$

and u_i goes linearly from 1 to 0 along the passages P_{i-1} and P_i.

Denote by $Q^\varepsilon(u)$ the quotient in the definition of $\Gamma^\varepsilon(\Omega)$. Notice that

a) $Q^\varepsilon(u) \leq 1$ for every $u \in L_1^2(\Omega)$; $u \neq 0$

and b) $\Gamma^\varepsilon \leq 1$ for every $\varepsilon > 0$.

In order to estimate $Q^\varepsilon(u)$ for our function u_i, notice that

$$\iint_{\Omega^\varepsilon} u_i^2 = \iint_{\Omega} u_i^2$$

and that

$$\iint_{\Omega} u_i^2$$

can be approximated by

$$\iint_{R_i} u_i^2$$

by neglecting terms of order higher than η_i^2 ,
and that

$$\operatorname{supp} \operatorname{grad}^2 u_i = P_{i-1} \cup P_i \ .$$

Then

$$Q^{\varepsilon}(u_i) = \frac{\displaystyle\iint_{R_i} 1^2}{\displaystyle\iint_{R_i} 1^2 + \iint_{P_{i-1}} \operatorname{grad}^2 u_i + \iint_{P_i} \operatorname{grad}^2 u_i}$$

$$= \frac{\eta_i^2}{\eta_i^2 + (1/\eta_i)^2\, \eta_i\, \eta_i^{3+\alpha} + (1/\eta_{i+1})^2\, \eta_{i+1}\, \eta_{i+1}^{3+\alpha}}$$

$$= \frac{1}{1+\eta_i^{\alpha} + (\eta_{i+1}/\eta_i)^2\, \eta_{i+1}^{\alpha}} \ .$$

Hence, by choosing a convenient relationship between η_i and η_{i+1}, and a convenient value of α, we see that, as $\eta_i \to 0$ (i.e. as $\varepsilon \to 0$) we can obtain an arbitrary value of $\Gamma^{\varepsilon}(\Omega)$ between 1/2 and 1 , <u>including 1</u> !

Mazya [8] deals with similar types of regions Ω , as seen in figure (4), they are rooms of different decreasing size, adjoined by passages to a larger room.

The alteration in the shape of passages, as these become smaller and smaller, plays the same fundamental role as in Amick's work.

Referring to the "ubiquitous rooms-and-passages", Edmunds [7] comments that the boundaries of these counterexample regions are of class C everywhere except in <u>one</u> single point: the point of accumulation of the different rooms.

Notice that, if $\partial\Omega$ is a fractal, then $\partial\Omega$ is not of class C <u>even for one point</u>.

Nevertheless, the property of rooms and passages that allows

$$\Gamma^{\varepsilon}(\Omega) \not\to 0 \text{ as } \varepsilon \to 0$$

is a dramatic alteration in the shape of the passages--i.e. the proportion of their width vis a vis their length--when reducing their size.

The property of self similarity of shape, irrespective of size, that we formulated for our Ω , $\partial\Omega$ a fractal curve, sharply separates our regions from the counterexamples to Rellich 's theorem known in literature.

A word now on the Sobolev spaces quoted in the Rellich theorem: when discussin the theorem of Rellich, different authors work with (apparently) different Sobolev spaces: $H^{m,p}(\Omega)$, $W^{m,p}(\Omega)$, $L_k^p(\Omega)$, W_p^k , L_p^k ...

These spaces may be defined in a quite different way, e.g.: If Ω is an arbitrary domain, then

$H^{m,p}$ = the completion of $\{ u \in C^m(\Omega) \mid \| u \|_{m,p} < \infty \}$

with respect to the norm $\| \cdot \|_{m,p}$,
whereas

$$W^{m,p}(\Omega) = \{ u \in L^p(\Omega) \mid D^\alpha(u) \in L^p(\Omega) \text{ for } 0 \leq |\alpha| \leq m \} \; ,$$

where $D^\alpha(u)$ is the weak (or distributional) partial derivative; and

$$\| u \|_{m,p} = \{ \sum_{0 \leq |\alpha| \leq m} \| D^\alpha u \|^p_{L^p(\Omega)} \}^{1/p} \, .$$

But Meyers and Serrin [9] have proved that the last two spaces are equal, ... we will use the oldest notation: $L_1^2(\Omega)$, and we will understand it to be the completion of $C^1(\Omega)$ functions with respect to the $\| \cdot \|_{1,2}$.

Appendix to Chapter 0

It is perhaps interesting to notice that there exist other types of counterexamples to Rellich 's Theorem which are not of the "rooms and passages" type.

The region Ω in figure (1) is an infinite sequence of disjoint sets S_j of decreasing size, non-rampantly added to a big square room R of side $A > 2$.

The sets S_j have only one accumulation point in ∂R .

Each set S_j is the non-rampant sum of j^5 big square rooms R^i_j and j^5 small square rooms r^i_j .

The big rooms alternate with the small ones.

The length of the side of R^i_j is $A^i_j = A^1_j / i^2$ for $i \geq 2$; $A^1_j = 1/j^2$.

The length of the side of the smaller room r^i_j is

$$a^i_j = A^1_j/(i+1)^3 \text{ for } i \geq 1 .$$

Then,

$$\sum_{i=1}^{\infty} A^i_j + \sum_{i=1}^{\infty} a^i_j =$$

$$A^1_j \{ 1+\sum_{i=2}^{\infty} \frac{1}{i^2} + \sum_{i=1}^{\infty} \frac{1}{(i+1)^3} \} \leq 2 A^1_j = 2 \frac{1}{j^2} \qquad (1)$$

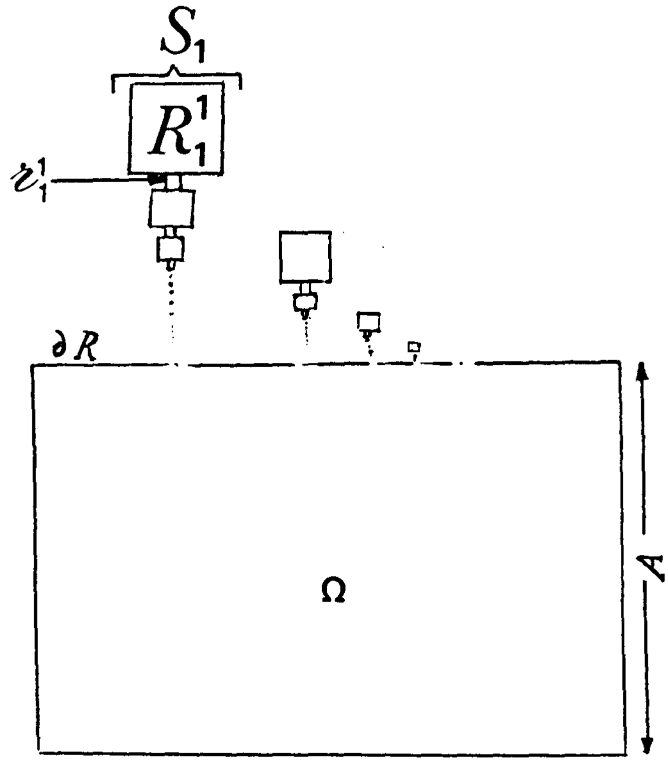


Figure 1

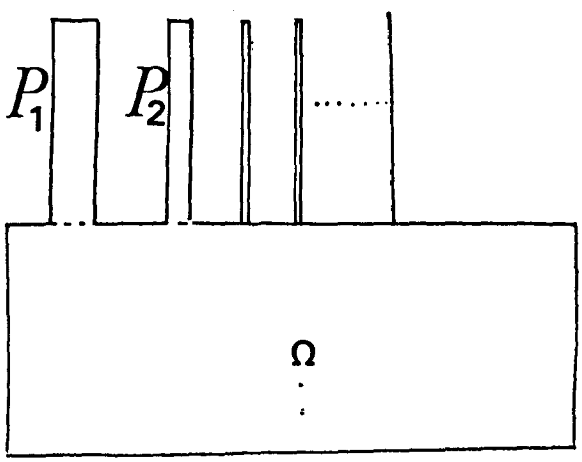


Figure 2

Let $\varepsilon > 0$ be <u>arbitrary</u>. Then, choosing j such that

$$1/j^2 < \varepsilon/2$$

we have $\mathcal{S}_j \subset \Omega^\varepsilon$.

Choose now a function $u \in L^2{}_1(\Omega)$ such that:

$\operatorname{supp} u_j \subset \mathcal{S}_j$; $u_j \equiv 1$ in $\mathcal{R}_j^1$,

$u_j \equiv \dfrac{j^5 - i}{j^5}$ in $\mathcal{R}_j^i$, and

u_j goes linearly from $\dfrac{j^5 - i}{j^5}$ to $\dfrac{j^5 - (i+1)}{j^5}$ on r_j^i

Then

$$\iint_{r_j^i} \operatorname{grad}^2 u_j = \left[\frac{\dfrac{j^5 - i}{j^5} - \dfrac{j^5 - (i+1)}{j^5}}{a_j^i} \right]^2 \mu^2(r_j^i)$$

$$= \frac{[\frac{1}{j^5}]^2}{(a_j^i)^2} (a_j^i)^2 = [\frac{1}{j^5}]^2$$

Inequality (1) implies that we can approximate

$$\iint_{\mathcal{S}_j} u_j^2 \quad \text{by} \quad \iint_{\mathcal{R}_j^1} u_j^2$$

by neglecting terms of order higher than $(A_j^1)^2$.

Also notice that

$$\iint_{\Omega} u^2 = \iint_{\Omega^\varepsilon} u^2$$

Therefore,

$$Q^{\varepsilon}(u_j) = \frac{\iint_{R_j^1} 1^2}{\iint_{R_j^1} 1^2 + j^5 [\frac{1}{j^5}]^2} = \frac{[\frac{1}{j^2}]^2}{[\frac{1}{j^2}]^2 + \frac{1}{j^5}}$$

$$= \frac{1}{1 + \frac{1}{j}} \to 1 \quad \text{when } j \to \infty \ .$$

Therefore

$$\Gamma^{1/j^2}(\Omega) \geq 1/(1+1/j)$$

and then

$$\Gamma^{\varepsilon}(\Omega) \to 1 \quad \text{when } \varepsilon \to 0 \ .$$

This last example can be called " rooms and rooms ". The next one can be called " passages and passages " :

As seen in figure (2) we adjoin disjoint passages P_j of width w_j , non rampantly to a large square room R of side of length A .

$$w_j \to 0 \quad \text{when } j \to \infty \ ;$$

$$\sum_{j=1}^{\infty} w_j < A \ ;$$

the length of all passages is 1 .

Given $\varepsilon > 0$, arbitrary, choose $j \in \mathbb{N}$ such that

$w_j < \varepsilon$, then $P_j \subset \Omega^\varepsilon$.

Now choose

$$u_j \in L^2_1(\Omega)$$

such that

$$\operatorname{supp} u_j \subset P_j ,$$

and u_j goes linearly from one to zero along the passage P_j .

As before we have :

$$\iint_{\Omega^\varepsilon} u_j^2 = \iint_{\Omega} u_j^2$$

then :

$$Q^\varepsilon(u_j) = \frac{w_j \int_0^1 x^2\,dx}{w_j \int_0^1 x^2\,dx + 1^2\, w_j}$$

$$= \frac{\frac{1}{3}}{\frac{1}{3} + 1} = \frac{1}{4}$$

Therefore $\Gamma^\varepsilon(\Omega)$ does not tend to zero when $\varepsilon \to 0$.

As in the case of rooms and passages already described, the fundamental alteration of proportions obtained with a change of scale plays the same crucial role in these other examples.

We will see in Chapter 4 a more precise way of separating all these cases from our Ω, $\partial\Omega$ a fractal curve.

Chapter 1

In this section we show how natural is the concept of extending the theorem of Rellich for regions Ω, $\partial\Omega$ a fractal curve.

Notation :

We will work many times with estimates of the type

$$c_1 B \leq A \leq c_2 B,$$

$$A \in \mathbf{R}^+, B \in \mathbf{R}^+, c_1 > 0, c_2 > 0,$$

where C_1 and C_2 are either absolute constants or constants depending solely on Ω. We denote the inequality above by the expression

$$A \sim B.$$

From now on, the generic letter c will denote any constant, either absolute or depending solely on Ω.

Let us recall that

$$\Gamma^{\xi}(\Omega) = \sup_{\substack{u \in L_1^2(\Omega) \\ u \neq 0}} Q^{\xi}(u)$$

where

$$Q^{\varepsilon}(u) = \frac{\iint_{\Omega^{\varepsilon}} u^2}{\iint_{\Omega} u^2 + \iint_{\Omega} \operatorname{grad}^2 u}$$

It has been proved [10] that, for $\partial\Omega$ sufficiently smooth we have:

$$\Gamma^{\varepsilon}(\Omega) \leq c_{\varepsilon} \qquad (1)$$

On the other hand, if we choose a constant function

$$u \equiv c,$$

then

$$Q^{\varepsilon}(u) = Q^{\varepsilon}(c) = \frac{\iint_{\Omega^{\varepsilon}} c^2}{\iint_{\Omega} c^2} = \frac{\mu^2(\Omega^{\varepsilon})}{\mu^2(\Omega)}$$

If $\partial\Omega$ is C^1 smooth and ε is small enough, then

$$\mu^2(\Omega^{\varepsilon}) \sim \mu^1(\partial\Omega)\,\varepsilon \qquad (2)$$

therefore

$$Q^{\varepsilon}(u) = c(\Omega)\,\varepsilon$$

and

$$\Gamma^{\varepsilon}(\Omega) = \sup_{\substack{u \in L_1^2(\Omega) \\ u \neq 0}} Q^{\varepsilon}(u) \geq c(\Omega)\,\varepsilon$$

Therefore, the estimates (1) and (2) imply

$$\Gamma^{\varepsilon}(\Omega) \sim \varepsilon$$

Hence, in order to calculate $\Gamma^{\varepsilon}(\Omega)$, where $\partial\Omega$ is a fractal, it becomes necessary to estimate first $\mu^2(\Omega^{\varepsilon})$.

Notice that, in estimate (2) we are given the dependence of $\mu^2(\Omega^{\varepsilon})$ both on ε and on $\partial\Omega$.

Now we will extend estimate (2) together with these dependences to the case in which $\partial\Omega$ is a fractal curve.

In order to specify the dependence on $\partial\Omega$ we choose the subset of $\partial\Omega$ generated by <u>one</u> single segment of $\partial\Omega^0$; i.e. the part of $\partial\Omega$ obtained by the iteration of the (n,N) process taking place in only one segment s of $\partial\Omega^0$.

Let Δ be the length of that segment s; for simplicity let us assume $\Delta < 1$, and let $\Omega^{\varepsilon}s$ denote the corresponding subset of Ω^{ε} associated with the segment s.

Lemma :

If $\varepsilon < \Delta$, then

$$\mu^2(\Omega^{\varepsilon}s) \sim \Delta^d \varepsilon^{2-d}$$

Note :

The lemma implies

$$\mu^2(\Omega^{\varepsilon}) \sim \mu^d(\partial\Omega)\,\varepsilon^{2-d} \qquad (3)$$

and is an extension of estimation (2) , obtained from (3) by setting $d = 1$.

Notice also the dimensionality of the exponents in the product $\Delta^d \varepsilon^{2-d}$: their sum is exactly 2 .

Proof :

Let $0 < \varepsilon < 1$. We begin by showing it suffices to prove the lemma for $\varepsilon = \Delta/n^p$. So suppose the lemma proved for $\varepsilon = \Delta/n^p$; for every $p \in \mathbb{N}$. Given $\varepsilon > 0$, choose $p \in \mathbb{N}$ such that

$$\Delta/n^{p+1} \le \varepsilon \le \Delta/n^p$$

Then

$$c_1 \Delta^d \varepsilon^{2-d} = 1/n^{2-d} \Delta^d \varepsilon^{2-d} \le 1/n^{2-d} \Delta^d (\Delta/n^p)^{2-d}$$

$$= \Delta^d (\Delta/n^{p+1})^{2-d} \sim \mu^2(\Omega^{\Delta/n^{p+1}} \mathcal{S}) \le \mu^2(\Omega^{\varepsilon} \mathcal{S})$$

$$\le \mu^2(\Omega^{\Delta/n^p} \mathcal{S}) \sim \Delta^d (\Delta/n^p)^{2-d} = n^{2-d} \Delta^d (\Delta/n^{p+1})^{2-d}$$

$$= c_2 \Delta^d (\Delta/n^{p+1})^{2-d} \le c_2 \Delta^d \varepsilon^{2-d} ,$$

whence

$$\mu^2(\Omega^{\varepsilon} \mathcal{S}) \sim \Delta^d \varepsilon^{2-d}$$

It remains only to prove the lemma for

$$\varepsilon = \Delta/n^p , \ p \in \mathbb{N} .$$

Now, there exists $q \in \mathbb{N}$ such that

$$1/n^{q+1} \le \Delta \le 1/n^q .$$

We reduce this general case to the particular one in which

$$\Delta = 1/n^q$$

by following the method above.

Let then

$$\Delta = 1/n^q \ , \ \varepsilon = \Delta/n^p = 1/n^{p+q} \ .$$

Let us suppose the result proved for

$$(\Omega^p s)^\varepsilon \ .$$

By neglecting subsets whose measure is of higher order than $\varepsilon^{2-d}\Delta^d$ we can write $\Omega^\varepsilon s$ as a non rampant sum of $(\Omega^p s)^\varepsilon$ and tails of decreasing size :

$$\Omega^\varepsilon s = (\Omega^p s)^\varepsilon +$$

$$+ \sum_{i=1}^{N^p} T_i^{p+1} + \sum_{i=1}^{N'N^p} T_i^{p+2} + \sum_{i=1}^{N'^2N^p} T_i^{p+3} + \ldots$$

where $N' < n$.

In the case of the Koch snowflake we have :

$$N' = 2 \ , \ N = 4 \ , \ n = 3 \ .$$

Then

$$\mu^2(\Omega^\varepsilon s) = \mu^2 [(\Omega^p s)^\varepsilon] +$$

$$+ \sum_{i=1}^{N^p} \mu^2(T_i^{p+1}) + \sum_{i=2}^{\infty} \quad \sum_{j=1}^{(N')^{i-1}N^p} \mu^2(T_j^{p+i})$$

Now

$$\mu^2(T^k) = c\,(\Delta/n^k)^2 \ ,$$

therefore

$$\sum_{i=1}^{N^p} \mu^2(T_i^{p+1}) + \sum_{i=2}^{\infty} \sum_{j=1}^{(N')^{i-1}N^p} \mu^2(T_j^{p+i}) =$$

$$c N^p (\Delta/n^{p+1})^2 + c \sum_{i=2}^{\infty} (N')^{i-1} N^p (\Delta/n^{p+i})^2 =$$

$$c \Delta^2 N^p (1/(n^{p+1})^2 + \sum_{i=2}^{\infty} \frac{(N')^{i-1}}{(n^{p+i})^2}) =$$

$$c \Delta^2 N^p 1/n^{2p} [1/n^2 + 1/N' \sum_{i=2}^{\infty} \frac{(N')^i}{n^{2i}}] =$$

$$c \Delta^d \Delta^{2-d} n^{dp} 1/n^{2p} K =$$

$$c \Delta^d (\Delta/n^p)^{2-d} K = c \Delta^d \varepsilon^{2-d} K$$

where

$$K = 1/n^2 + 1/N' \sum_{i=2}^{i=\infty} \frac{(N')^i}{n^{2i}}$$

is bigger than $1/n^2$ and smaller than

$$1/n^2 + 1/N' \sum_{i=2}^{\infty} \frac{N^i}{n^{2i}}$$

which is finite, since $N < n^2$.

We only have to prove

$$\mu^2 [(\Omega^p s)^\varepsilon] \sim \Delta^d \varepsilon^{2-d}$$

and this is immediate :

if $\varepsilon = \Delta/n^p = 1/n^{p+q}$ then, each segment of $\partial\Omega^p s$ has length equal to $\Delta/n^p = 1/n^{p+q} = \varepsilon$, hence, the contribution of <u>each</u>

of these segments to the area of $(\Omega^p s)^\varepsilon$ is $c\,\varepsilon^2$; where $c = c(\Omega)$ can have at most N different values ; therefore

$$\mu^2 [\,(\Omega^p s)^\varepsilon\,] \sim N^p\,\varepsilon^2 =$$

$$\frac{n^{dp}}{n^{2p}\,n^{2q}} = \frac{n^{dp}}{n^{2p}\,n^{(2-d)q}\,n^{dq}} = \left(\frac{1}{n^q\,n^p}\right)^{2-d}\left(\frac{1}{n^q}\right)^d = \varepsilon^{2-d}\,\Delta^d \text{ , q.e.d.}$$

Chapter II

In the first proofs of the classical theorem of Rellich, and of other theorems in analysis, the condition of smoothness for the boundary $\partial\Omega$ of the bounded region Ω was C^1.

That condition is satisfied by fractal curves at no point whatsoever; neither is condition class C.

Subsequent proofs slightly relaxed, sometimes, this C^1 condition on $\partial\Omega$, for different theorems, down to class C, the cone property or the segment property.

Now, we can certainly find many points in a fractal curve $\partial\Omega$ each one being the vertex of a cone contained in Ω, and many more points which are the end points of a segment contained in Ω.

However, we will prove that these points in $\partial\Omega$ constitute a set of Hausdorff d-measure zero.

Remark 1:

Notice that the (n, N) process of replacement, iterated *ad infinitum*, implies the existence, in $\partial\Omega$, of horns with an infinity of turns. See figure (1) for the Koch snowflake.

These horns are dense in $\partial\Omega$.

Let $P = \{ x \in \partial\Omega \ / \ \partial\Omega \text{ meets some neighbourhood of } x \text{ in a } C^1 \text{ curve} \}$

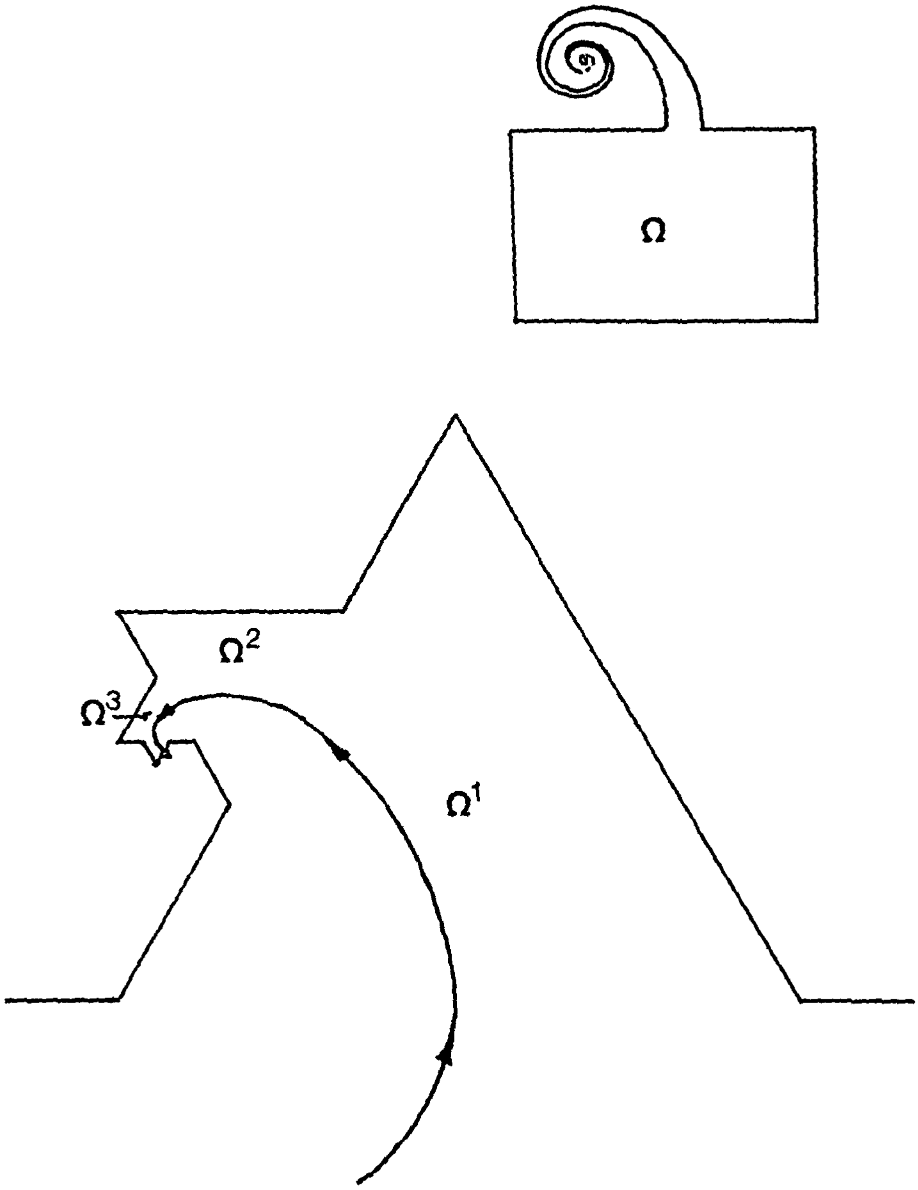


Figure 1

$$Q = \{ x \in \partial\Omega \ / \ \exists\, c^{\theta}(x) \subset \Omega \ , \theta \in (0,2\pi) \ \}$$
$$R = \{ x \in \partial\Omega \ / \ \exists\, s^{\rho}(x) \subset \Omega \ , \rho > 0 \ \}$$
where $c^{\theta}(x)$ is a finite cone with vertex x and angle θ ; and $s^{\rho}(x)$ is a segment with endpoint x and length ρ .

We have : $P \subset Q \subset R$.

Denote : $\mathbf{P} = \partial\Omega - P$; $\mathbf{Q} = \partial\Omega - Q$; $\mathbf{R} = \partial\Omega - R$; we have, then : $\mathbf{P} \supset \mathbf{Q} \supset \mathbf{R}$; and
$$\mu^{d}(\partial\Omega) = \mu^{d}(\mathbf{P}) \geq \mu^{d}(\mathbf{Q}) \geq \mu^{d}(\mathbf{R})$$

Theorem :
$$\mu^{d}(\mathbf{R}) = \mu^{d}(\partial\Omega)$$

Proof :

If $\quad \mathbf{R}^{\rho} = \{ x \in \partial\Omega / \nexists \ s^{\rho}(x) \subset \Omega \}$

then $\mathbf{R} = \liminf \mathbf{R}^{\rho}$ when $\rho \to 0$.

Therefore
$$\mu^{d}(\mathbf{R}^{\rho}) \to \mu^{d}(\mathbf{R}) \quad \text{when} \quad \rho \to 0 \ .$$
Hence it is enough to prove that
$$\mu^{d}(\mathbf{R}^{\rho}) = \mu^{d}(\partial\Omega) \quad \text{for every} \ \rho > 0 \ .$$

Let us recall that T^{p} is a tail of size p , $p \in \mathbf{N}$. Given ρ , choose $p \in \mathbf{N}$ so big as to ensure
$$\text{diam}\,(T^{p}) < \rho/2 \ .$$

For each segment $s_{k}{}^{p} \subset \partial\Omega^{p}$; $\mu^{1}(s_{k}{}^{p}) = 1/n^{p}$
$$k \in \{ 1,2,\ldots\ldots,N^{p} \}$$
we consider the $\partial\Omega^{p+i}$ associated with it -i.e.obtained by

iterating the (n,N) process i times in s_k^p, $i \in \mathbb{N}$, and we denote it by $\partial\Omega_k^{p+i}$.

Because of remark (1), there exists an $i \in \mathbb{N}$ and a segment

$$s_{k;0}^{p+i} \subset \partial\Omega_k^{p+i} \quad ; \quad \mu^1(s_{k;0}^{p+i}) = \frac{1}{n^{p+i}}$$

such that there is no

$$x \in s_{k;0}^{p+i}$$

with the property:

$$s^p(x) \subset \Omega \ .$$

We can find this

$$s_{k;0}^{p+i} \subset \partial\Omega_k^{p+i}$$

by searching for it deep enough inside a horn in $\partial\Omega_k^{p+j}$, $j \in \mathbb{N}$.

For instance, for the Koch snowflake, as seen in figure (2) we have $i = 3$; in $\partial\Omega_k^{p+3}$ there appear 2 such segments that can be choosen to be $s_{k;0}^{p+3}$; of length $1/n^{p+3}$; and, if

$$x \in s_{k;0}^{p+3}$$

then any $s^p(x)$ is not contained in Ω.

Remark 2 :

Because of the regularity of the (n,N) process, this integer i can be choosen independently of the particular choice of s_k^p, $k \in \{1,2,\ldots,N^p\}$.

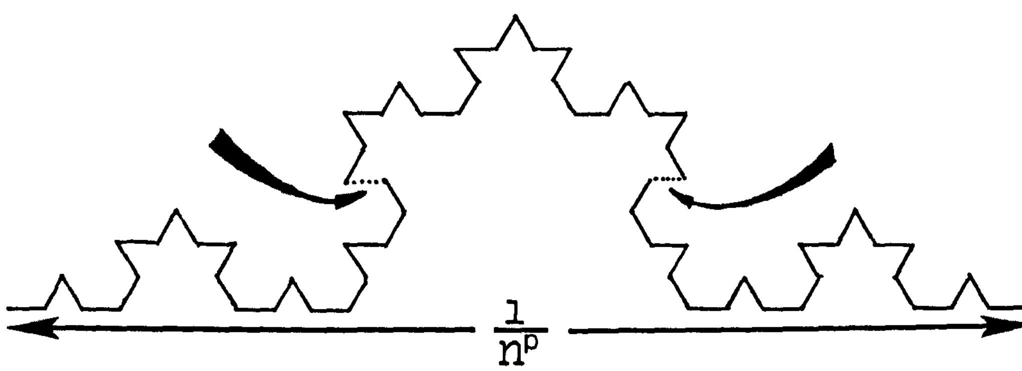


Figure 2

Denote that part of $\partial\Omega$ associated with $\mathcal{S}^{p+i}_{k;0}$ – i.e. obtained by iterating the (n,N) process *ad infinitum* in $\mathcal{S}^{p+i}_{k;0}$ – by $\partial^{p+i}_{k;0}\Omega$.

We have :

a$_1$) $\partial^{p+i}_{k;0} \subset \mathbb{R}^{\rho}$, and

a$_2$) $\mu^d(\partial^{p+i}_{k;0}) = \dfrac{\mu^d(\partial^p_k\Omega)}{N^i}$

Let us take now <u>any</u> other segment

$$\mathcal{S}^{p+i}_{k;j} \neq \mathcal{S}^{p+i}_{k;0} \quad j \in \{1,2,\ldots,N^p\}$$

$$\mathcal{S}^{p+i}_{k;j} \subset \partial\Omega^{p+i}_k \ ; \ \mu^1(\mathcal{S}^{p+i}_{k;j}) = \frac{1}{n^{p+i}}$$

and consider the part of $\partial\Omega^{p+2i}$ associated with it, that is, consider $\partial\Omega_{k;j}{}^{p+2i}$.

Because of remark (2) there is at least one segment

$$\mathcal{S}^{p+2i}_{k;j;0} \text{ in } \partial\Omega^{p+2i}_{k;j} \ ;$$

$$\mu^1(\mathcal{S}^{p+2i}_{k;j;0}) = \frac{1}{n^{p+2i}}$$

such that, if $x \in \mathcal{S}^{p+2i}_{k;j;0}$, then there is no $s^\rho(x)$ contained in Ω .

On the corresponding associated part of $\partial\Omega$, we know :

b$_1$) $\partial^{p+2i}_{k;j;0}\,\Omega \subset \mathbb{R}^{\rho}$, and

b$_2$) $\mu^d(\partial^{p+2i}_{k;j;0}\,\Omega) = \dfrac{\mu^d(\partial^p_k\Omega)}{N^{2i}}$

Notice that this process takes place for $N^i - 1$ values of j, that is : we have $N^i - 1$ of such sets

$$\partial^{p+2i}_{k;j;0}\,\Omega \subset \mathbf{R}^p$$

If we iterate the process once more we will obtain $N^i - 1$ sets

$$\partial^{p+3i}_{k;j;l;0}\,\Omega$$

for each of the $N^i - 1$ values of j just referred to, that is : we will obtain $(N^i - 1)^2$ sets

$$\partial^{p+3i}_{k;j;l;0}\,\Omega$$

contained in $\mathbf{R}^p$ and of d-measure

$$\frac{\mu^d(\partial^p_k\Omega)}{N^{3i}}$$

and so on ...

Using equalities $a_2)$, $b_2)$, and the last remark, we can now calculate $\mu^d(\, \mathbf{R}^p \cap \partial_k{}^p\Omega\,)$:

$$\mu^d(\, \mathbf{R}^p \cap \partial^p_k\Omega\,) \geq \sum_{l=1}^{\infty} \frac{\mu^d\,(\,\partial^p_k\,\Omega)}{N^{li}}\,(\,N^i - 1)^{l-1}$$

$$= \mu^d\,(\,\partial^p_k\,\Omega)\,\frac{1}{N^i}\,(\,1 + \frac{N^i - 1}{N^i} + \{\frac{N^i - 1}{N^i}\}^2 + \ldots)$$

$$= \mu^d\,(\,\partial^p_k\,\Omega)\,\frac{1}{N^i}\;\frac{1}{1 - \dfrac{N^i - 1}{N^i}} = \mu^d\,(\,\partial^p_k\Omega)\,\frac{1}{N^i}\,\frac{1}{\dfrac{1}{N^i}}$$

$$= \mu^d (\partial_k{}^p \Omega) \quad .$$

Therefore:

$$\mu^d(\mathbf{R}^p \cap \partial_k{}^p \Omega) = \mu^d(\partial_k{}^p \Omega)$$

and since this is true for all $k \in \{ 1, 2, ..., N^p \}$ we have finally:

$$\mu^d(\mathbf{R}^p) = \mu^d(\partial\Omega) \qquad \text{q.e.d.}$$

Chapter 3

The next result will help in clarifying the role of

$$\iint_{\Omega} u^2$$

in the denominator of

$$Q^{\varepsilon}(u) = \frac{\| u \|^2_{L^2(\Omega^{\varepsilon})}}{\| u \|^2_{L^2_1(\Omega)}} \quad .$$

Without it, we would be left with

$$\overline{Q^{\varepsilon}(u)} = \frac{\iint_{\Omega^{\varepsilon}} u^2}{\iint_{\Omega} \mathrm{grad}^2\, u} \quad ,$$

which, for functions u supported on the dangerous part Ω^{ε} (i.e. near the non-smooth fractal boundary), is exactly the quotient

$$\frac{\| u \|^2_{L^2}}{\| \mathrm{grad}\, u \|^2_{L^2}}$$

used for so many purposes in a diversity of problems in Analysis.

We will find a connection between our Q^{ε} and the "classical" $\overline{Q^{\varepsilon}}$: more specifically, we will write Ω as a non-rampant sum of a main body Ω^0 , and a finite number n_0 of tails

T_i , and we will reduce the general case of a function

$$u \in L_1^2(\Omega)$$

to the case of

$$u' \in L_1^2(\sum_{i=1}^{n_0} T_i) \ ,$$

and we will show in later chapters, that for such functions u' , it is enough to work with $\overline{Q^\epsilon}(u)$ instead of $Q^\epsilon(u)$, in order to obtain our results.

We need:

Lemma :

Let $(x,y) \leftrightarrow (u,v)$ be a 1:1 continuous mapping of the region Ω onto itself; $\partial\Omega$ is mapped onto itself; the jacobian of the transformation is bounded below away from zero, all coefficients in the matrix jacobian are bounded in absolute value.

Let

$$f \in L_1^2(\Omega) \ , \text{ and let } \bar{f}(u,v)$$

be the transformed function.

Then

$$\mathrm{grad}^2 \bar{f} \sim \mathrm{grad}^2 f \ ,$$

the constants of proportionality depending on the given mapping.

Proof. : It is quite straightforward. We have:

$$\overrightarrow{\mathrm{grad}\, \bar{f}} = \mathbb{J}\, \overrightarrow{\mathrm{grad}\, f} \ ,$$

where $\mathbf{J}$ is the matrix jacobian of the mapping.

The boundedness of the coefficients $\mathbf{J}$ implies

$$|\overrightarrow{\operatorname{grad} \bar{f}}| \le c\, |\overrightarrow{\operatorname{grad} f}| \quad , \text{ that is}$$

$$\operatorname{grad}^2 \bar{f} \le c \operatorname{grad}^2 f \quad .$$

The boundedness away from zero of the jacobian, plus the boundedness of the coefficients of $\mathbb{J}$, imply both the invertedness of $\mathbb{J}$ and the boundedness of the coefficients of $\mathbb{J}^{-1}$, whence

$$\operatorname{grad}^2 f \le c \operatorname{grad}^2 \bar{f} \quad , \text{ q. e. d.}$$

Now we can prove:

Theorem :

Let us write Ω as a non rampant sum of a main body Ω° and a finite number of tails $T^1_i = T_i$ of size 1 :

$$\Omega = \Omega^{\circ} + \sum_{i=1}^{n_0} T_i$$

Let the size of Ω be normalized so that $\mu^1(T_i \cap \Omega^{\circ}) = 1$.

Then, there exists a constant $c = c\,(\Omega)$ such that, for any $u \in L^2_1(\Omega)$, there exists $u' \in L^2_1(\Omega)$;

$$\operatorname{supp} u' \subset \sum_{i=1}^{n_0} T_i \quad .$$

and $Q^{\varepsilon}(u) \le c\, Q^{\varepsilon}(u')$

Proof :

Consider, in each tail T_i, a strip $S_i \subset T_i$ as shown in figure (1) for the case of the Koch snowflake.

The width of each S_i will be called Δ, and is a number independent of the index i. In figure (1)(b), Δ has been chosen

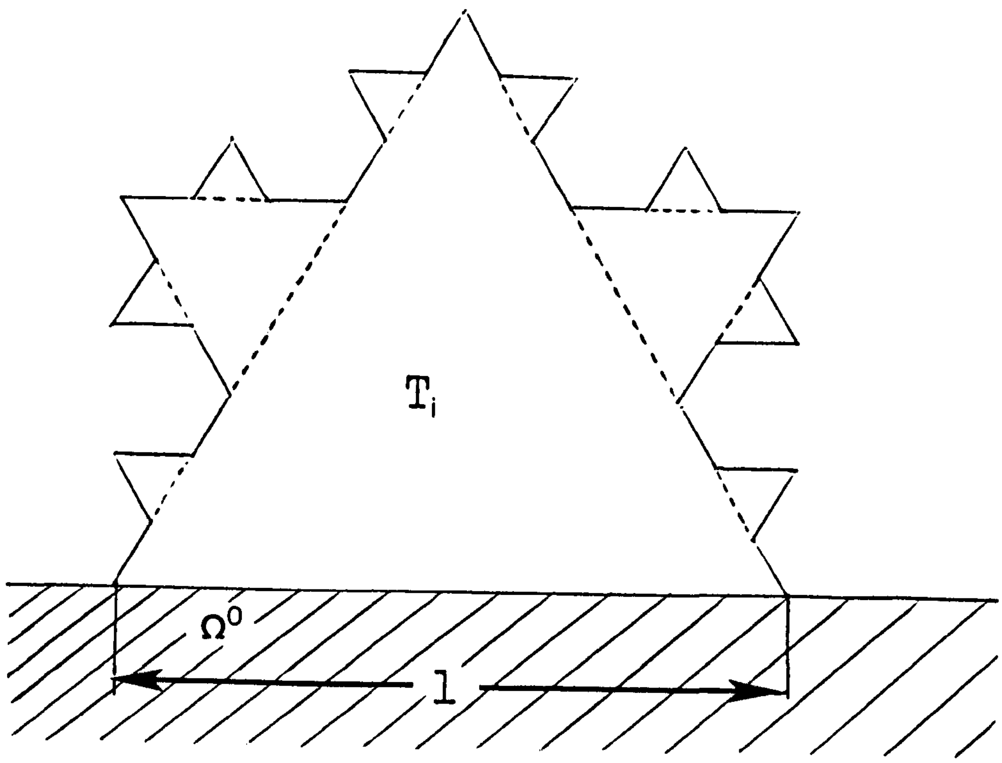


Figure 1(a)

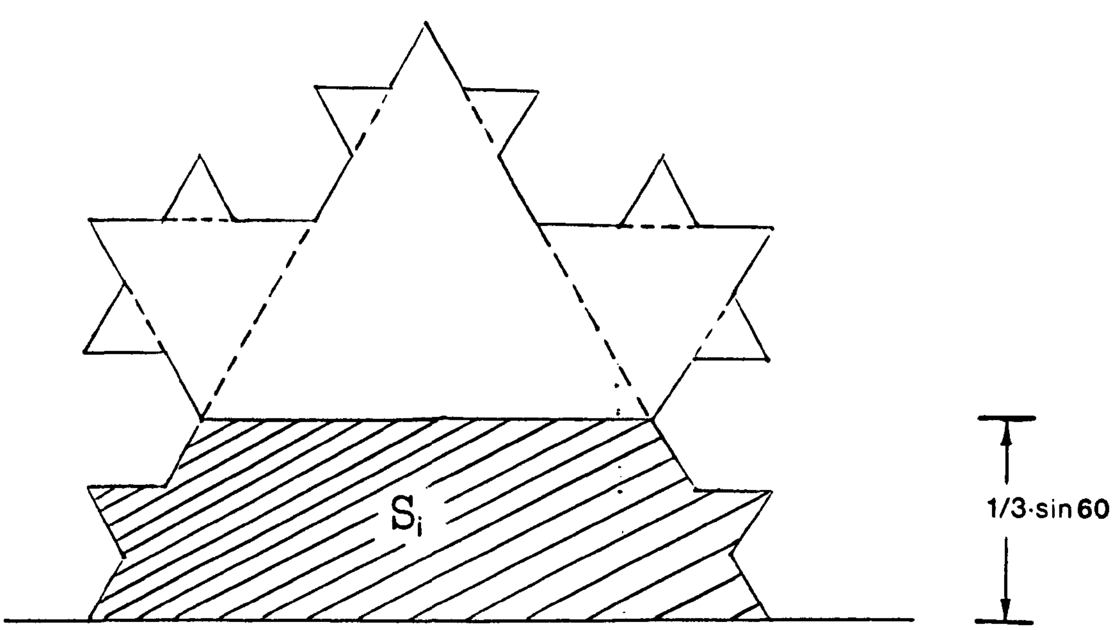


Figure 1(b)

to be $1/n \cdot \sin 60°$.

The number Δ must fulfill the following requirement :

$$\text{If} \quad A^\varepsilon = \Omega^\varepsilon \cap \{ \sum_{i=1}^{n_0} (T_i - S_i) \} \ , \ \text{then}$$

$\mu^2(A^\varepsilon) > 1/2 \, \mu^2 (\Omega^\varepsilon)$, that is $\Omega^\varepsilon - A^\varepsilon$ is smaller than A^ε .

Let $v_i \in L^2{}_1 (\Omega)$, supp $v_i = T_i$, be such that :

$v_i \equiv 1$ in $T_i - S_i$, and v_i goes linearly from one to zero in the strip S_i .

Next, let $v \in L^2{}_1 (\Omega)$ be supported in

$$\sum_{i=1}^{n_0} T_i \ .$$

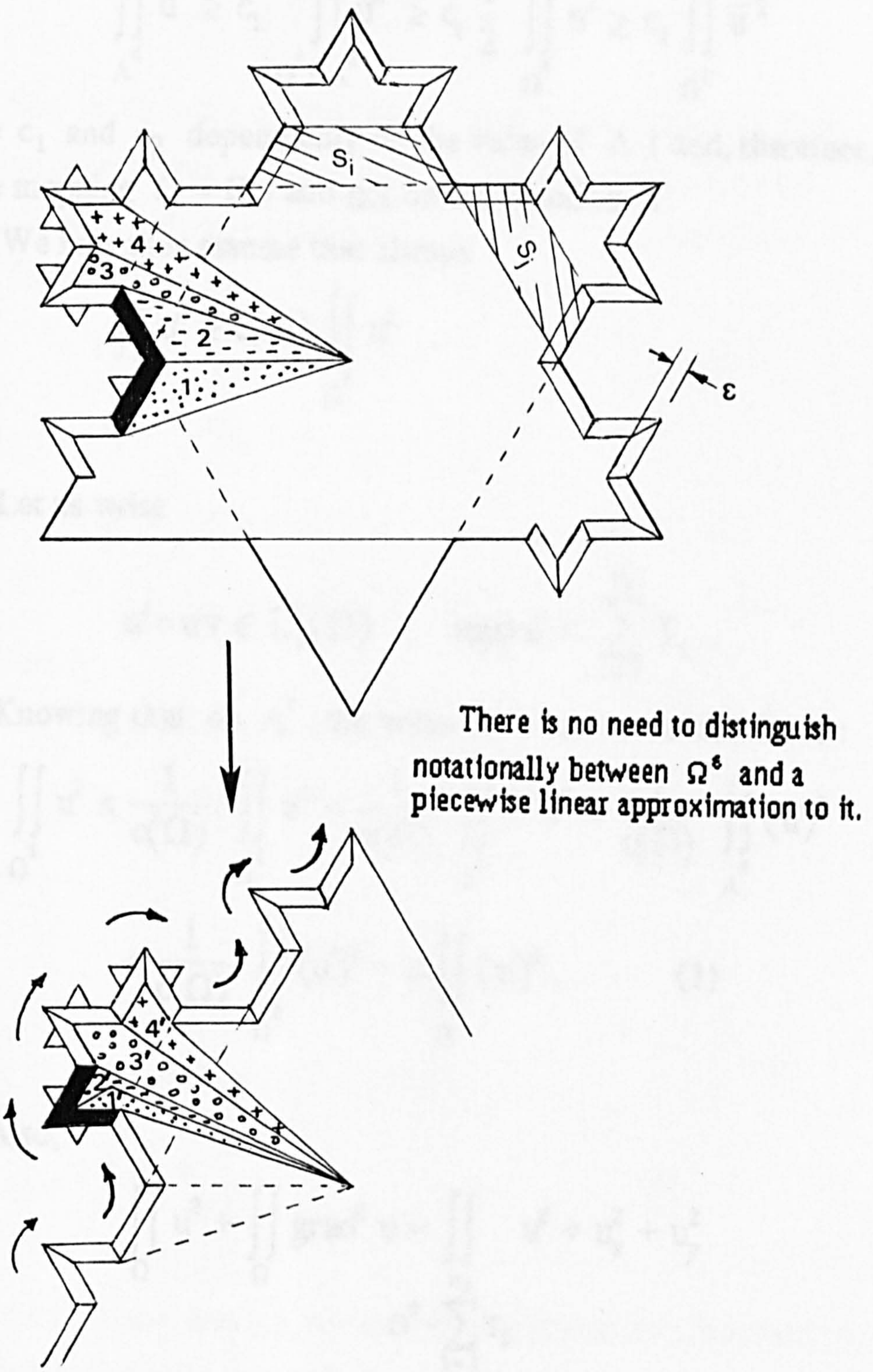
defined thus : $v = v_i$ in each T_i .

Now, let u be any function in $L^2{}_1 (\Omega)$.

$$\text{If} \quad \iint_{\Omega^\varepsilon - A^\varepsilon} u^2 > \frac{1}{2} \iint_{\Omega^\varepsilon} u^2 \ ,$$

then there is a continuous one to one transformation of Ω onto Ω , $\partial\Omega$ onto $\partial\Omega$, such that the jacobian satisfies the hypotheses of our lemma, which also mapps Ω^ε onto itself, and $\Omega^\varepsilon - A^\varepsilon$ into A^ε (because of our requirement on the value of Δ). There are many ways of effecting this transformation, but a kind of "rotation" of Ω^p onto itself with p=2 , shifting $(\Omega^2)^\varepsilon$ (and $\Omega \dot{-} \Omega^2$ being transformed onto itself in the natural way) of the type indicated in figure (2) will do : it merely shifts the position of Ω^ε .

If $\tilde{u}$ is the induced function, then by our lemma



S_i
S_j
ε
1
2
3
4
1'
2'
3'
4'
There is no need to distinguish notationally between Ω^ε and a piecewise linear approximation to it.

Figure 2

$$Q^{\varepsilon}(\bar{u}) \sim Q^{\varepsilon}(u) \text{ , and}$$

$$\iint_{A^{\varepsilon}} \bar{u}^2 \geq c_1 \iint_{\Omega^{\varepsilon}-A^{\varepsilon}} u^2 \geq c_1 \frac{1}{2} \iint_{\Omega^{\varepsilon}} u^2 \geq c_2 \iint_{\Omega^{\varepsilon}} \bar{u}^2$$

where c_1 and c_2 depend only on the value of Δ (and, therefore, on the mapping $\Omega \to \Omega$) and not on the function u.

We may then assume that always

$$\iint_{A^{\varepsilon}} u^2 \geq c(\Omega) \iint_{\Omega^{\varepsilon}} u^2 \quad .$$

Let us write

$$u' = uv \in L_1^2(\Omega) \quad ; \quad \operatorname{supp} u' \subset \sum_{i=1}^{n_0} T_i \quad .$$

Knowing that, on A^{ε}, the value of v is 1, we then have :

$$\iint_{\Omega^{\varepsilon}} u^2 \leq \frac{1}{c(\Omega)} \iint_{A^{\varepsilon}} u^2 = \frac{1}{c(\Omega)} \iint_{A^{\varepsilon}} u^2 v^2 = \frac{1}{c(\Omega)} \iint_{A^{\varepsilon}} (u')^2$$

$$\leq \frac{1}{c(\Omega)} \iint_{\Omega^{\varepsilon}} (u')^2 = c \iint_{\Omega^{\varepsilon}} (u')^2 \qquad (1)$$

Also,

$$\iint_{\Omega} u^2 + \iint_{\Omega} \operatorname{grad}^2 u = \iint_{\Omega^0 + \sum_{i=1}^{n_0} T_i} u^2 + u_x^2 + u_y^2$$

$$\geq \sum_{i=1}^{n_o} \iint_{T_i} u^2 + u_x^2 + u_y^2 \qquad (2')$$

Now let us focus on one tail T_i; and let us define the local cartesian coordinates $(\tilde{x},\tilde{y})$, where $\tilde{x}$ is parallel to the segment $\partial T_i \cap \partial\Omega^o$, as shown in figure (3) for the Koch snowflake.

Therefore, for a.e. $(\tilde{x},\tilde{y}) \in T_i$ we have:

$$(u')^2 = u^2 v^2 = u^2 v_i^2 \leq u^2 \qquad (a) \;;$$

$$u'_x = (uv)_x = u_x v_i + u\, v_{i_x}$$

but v_{i_x} is zero, then

$$(u'_x)^2 = u_x^2 v_i^2 \leq u_x^2 \quad (\text{since } v_i \leq 1) \qquad (b)$$

Finally,

$$u'_y = (u\,v)_y = u_y v_i + u\, v_{i_y} \quad ;$$

but v_{i_y} is either zero (in $T_i - S_i$), or $\frac{1}{\Delta}$ (in S_i),

therefore

$$v_{i_y}^2 \leq \frac{1}{\Delta^2} \quad , \text{ and then}$$

$$(u'_y)^2 \leq 2\,(u_y v_i)^2 + 2\,(u\, v_{i_y})^2$$

$$\leq 2\, u_y^2 + \frac{2}{\Delta^2} u^2 \qquad (c)$$

From (a), (b), and (c), we have that there exists a constant c, depending only on the value of Δ, such that

$$\iint_{T_i} (u')^2 + (u'_x)^2 + (u'_y)^2 \;\leq c \iint_{T_i} u^2 + u_x^2 + u_y^2 \;\; ; (c = 2 + \frac{2}{\Delta^2})$$

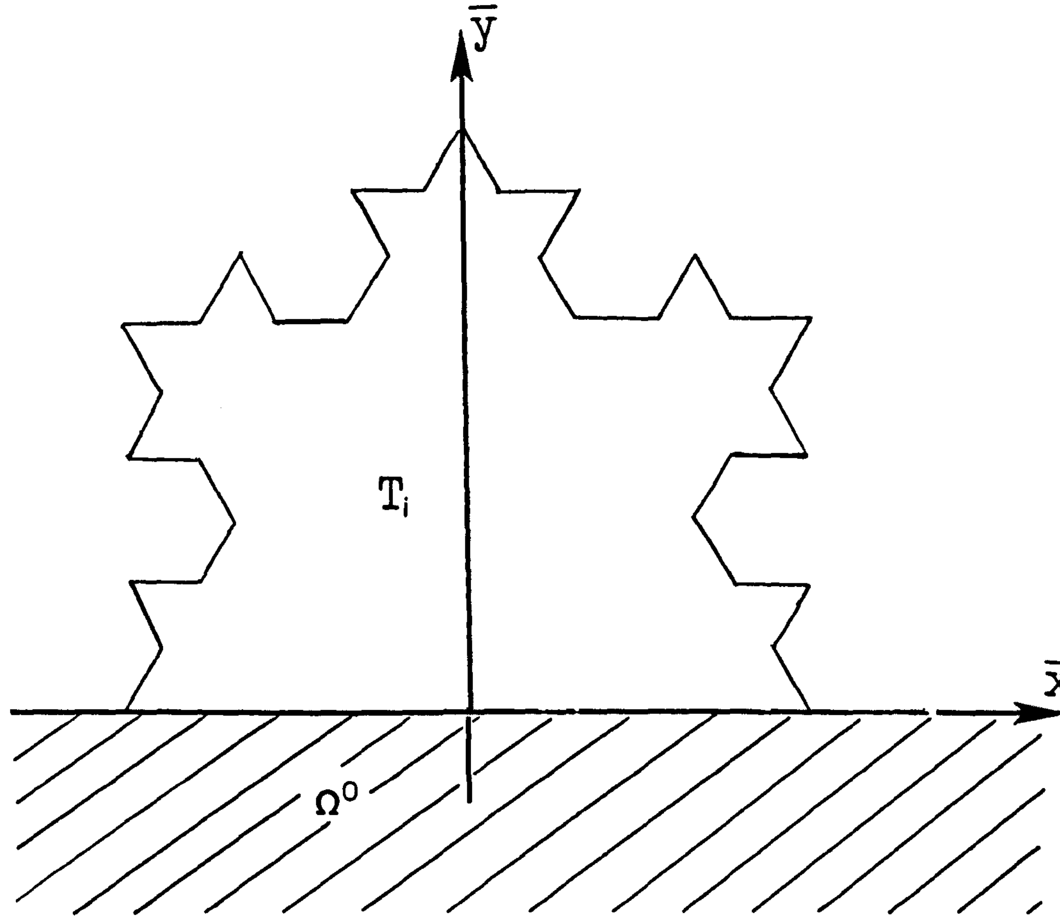


Figure 3

and then (2') can be rewritten as

$$\iint_\Omega u^2 + \iint_\Omega \mathrm{grad}^2\, u \geq \frac{1}{c} \sum_{i=1}^{n_o} \iint_{T_i} (u')^2 + (u'_x)^2 + (u'_y)^2$$

$$= \frac{1}{c} \{ \iint_\Omega u'^2 + \iint_\Omega \mathrm{grad}^2\, u' \} \ , \ (\text{since } \operatorname{supp} u' \subset \sum_{i=1}^{n_o} T_i \,) \quad (2)$$

From (1) and (2) we finally obtain

$$Q^\varepsilon(u) \leq c\, Q^\varepsilon(u') \ , \ \text{where } c = c(\Omega) \text{ only q.e.d.}$$

Therefore, we are reduced to proving

$$Q^\varepsilon (u') \leq c\, \varepsilon^{2-d}$$

when u' is supported on a <u>finite</u> number of tails.

We will prove that for such functions u' we have

$$\overline{Q}^\varepsilon (u') \leq c\, \varepsilon^{2-d}$$

and, since

$$Q^\varepsilon(u') \leq \overline{Q}^\varepsilon(u') \quad ,$$

our result will follow .

Another simplification: Since $L_1^2(\Omega)$ is the completion, in L_1^2 norm, of $C^1(\Omega)$; we can see that, in order to prove

$$\overline{Q}^\varepsilon (u) \leq c\, \varepsilon^{2-d}$$

for $u \in L_1^2(\Omega)$, it is enough to prove it for functions in $L_1^2(\Omega) \cap C^1(\Omega)$. Moreover, the same simplification applies when proving <u>each</u> of the results in the chapters that follow, since those results refer to norms of functions in $L_1^2(\Omega)$. Hence, throughout, when referring to a function in $L_1^2(\Omega)$, we will understand it to be in $C^1(\Omega)$ as well.

Notice that we can further restrict ourselves to positive functions.

Irregular Tails

The case analyzed above deals only with tails that are non rampant sums of self similar sets of decreasing size. A glance at figure (4) shows the possible existence of some tails, like the $A^i B^i C^i$ ones in that example, having their biggest triangle of different shape than all the others of decreasing size--i.e.

$$\overset{\Delta}{45°,45°,90°} \text{ as opposed to } \overset{\Delta}{60°,60°,60°}$$

We will call such a tail an irregular one.

However, this regularity does not introduce any fundamental difference in treatment vis-a-vis the case of a regular tail, since there is only one set with shape not self-similar to all the others of decreasing size; still, for the sake of completeness, we have to deal with such cases.

The chapters that follow prove theorems for regular tails.

One way of including the irregular ones is to write a little corollary at the end of each chapter, slightly modifying the argument in order to include that one set of shape not similar to the others.

However, there is a quick way of reducing the case of an irregular tail to the case of a regular one:

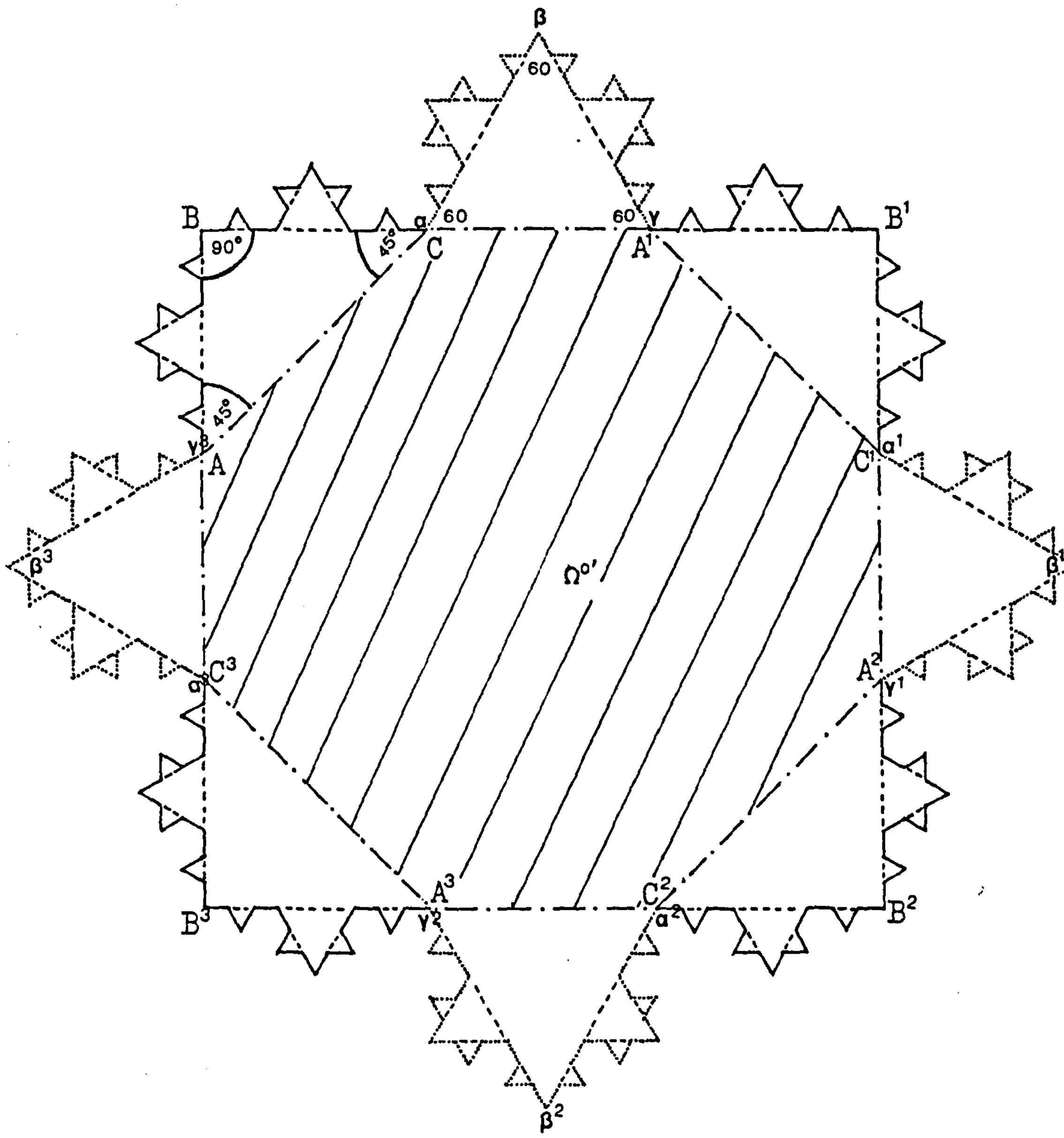

β
60
B
90°
45°
α
60
C
60
γ
A¹
B¹
45°
γ³
A
α¹
C¹
β³
Ω⁰′
β¹
C³
α³
A²
γ¹
A³
C²
B³
γ²
α²
B²
β²

Figure 4

Let us recall that the reasoning in the theorem takes us to the case in which u is supported on tails, and

$$\iint_{A^\varepsilon} u^2 \geq c \iint_{\Omega^\varepsilon} u^2 .$$

Let T_j, $j \in J$, be the irregular tails, and T_i, $i \in I$, the regular ones.

$$\text{Is} \quad \iint_{A^\varepsilon \cap \sum_{i \in I} T_i} u^2 \geq \iint_{A^\varepsilon \cap \sum_{j \in J} T_j} u^2 \quad ?$$

If the answer is "yes", we proceed just as in the theorem :

$$\overline{Q}^\varepsilon(u) = \frac{\iint_{\Omega^\varepsilon} u^2}{\iint_{\Omega} \operatorname{grad}^2 u} \leq c \frac{\iint_{A^\varepsilon} u^2}{\iint_{\sum_{i \in I} T_i} \operatorname{grad}^2 u + \iint_{\sum_{j \in J} T_j} \operatorname{grad}^2 u}$$

$$= c \frac{\iint_{A^\varepsilon \cap \sum_{i \in I} T_i} u^2 + \iint_{A^\varepsilon \cap \sum_{j \in J} T_j} u^2}{\iint_{\sum_{i \in I} T_i} \operatorname{grad}^2 u + \iint_{\sum_{j \in J} T_j} \operatorname{grad}^2 u} \leq 2c \frac{\iint_{A^\varepsilon \cap \sum_{i \in I} T_i} u^2}{\iint_{\sum_{i \in I} T_i} \operatorname{grad}^2 u}$$

$= 2c\,\overline{Q}^\varepsilon(\bar{u})$, where $\bar{u}$ is supported only on the <u>regular</u> tails .

If the answer is "no", then the relevant part of

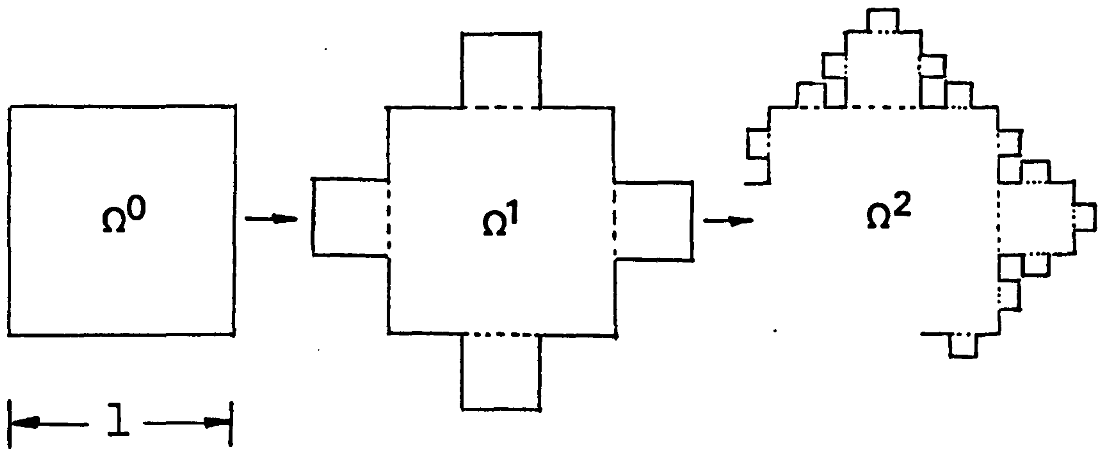
$$\iint_{\Omega^\varepsilon} u^2$$

is concentrated in the irregular tails, and, just as before in the theorem, we will shift the irregular tails, by means of a transformation (whose jacobian satisfies the hypotheses of our lemma) , into the regular ones , and then proceed as before.

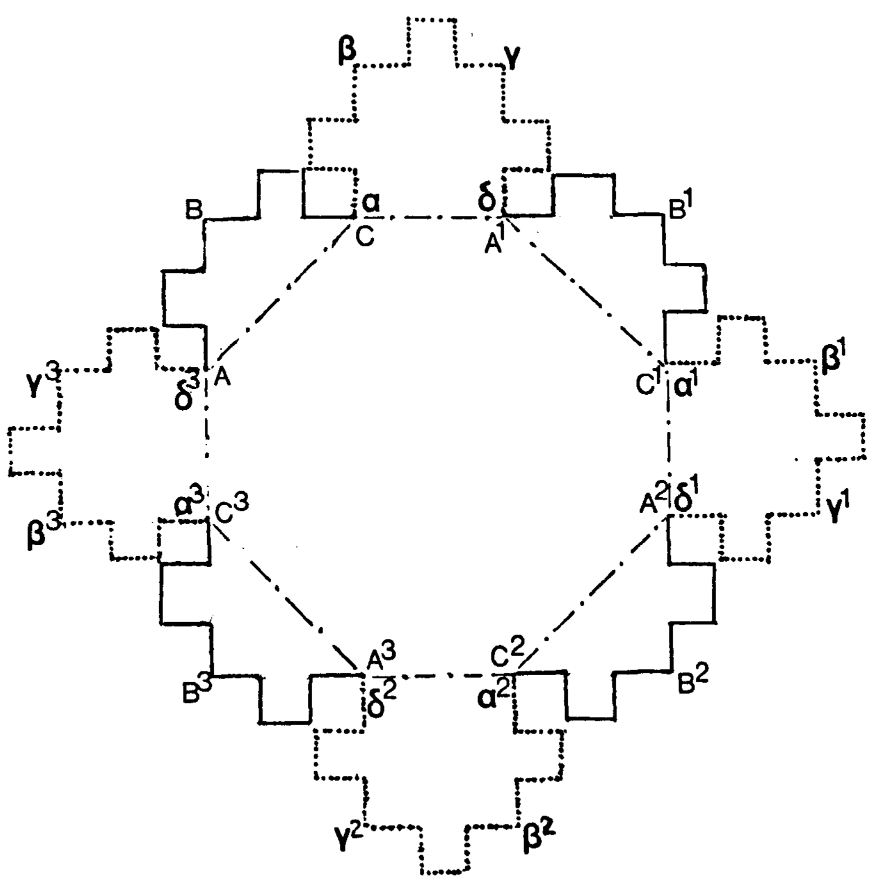
In the case of the process of replacement of the type $(n,N) = (3,4)$, as in the Koch snowflake, carried out in the four sides of a square, we can see, with just a glance at figure (4) , that a "rotation" of the type described in our theorem,will transform irregular tails $A^j B^j C^j$ in regular ones $\alpha^i \beta^i \gamma^i$, with the relevant set A^ε_j being transformed onto A^ε_i . But this case is not general, as an examination of the case depicted in figure (5) will immediately show. In that figure we can observe that there is no way in which any type of "rotation" of Ω^2 onto Ω^2 of the kind described above in our theorem, could ever carry the irregular tail ABC (whose boundary has two self-replicating sets of Hausdorff dimension $d = \lg 5 / \lg 3$ and size $(1/3)^d$) into the regular one $\alpha\beta\gamma\delta$ (whose boundary has three! such sets).

Still, there is a general way in which we can map Ω onto Ω , $\partial\Omega$ onto $\partial\Omega$, with jacobian as in the lemma, and mapping the relevant parts A^ε_j , $j \in J$, into the A^ε_i , $i \in I$.

As in the former case, it will suffice to indicate the equivalent of the already described relevant transformation of Ω^2 onto Ω^2 (which will not be that "rotation" any more), or rather, the equivalent of that relevant transformation of $\partial\Omega^2$ onto $\partial\Omega^2$



Ω0
Ω1
Ω2
1



β
γ
B
a
C
δ
A1
B1
γ3
δ3
A
C1
a1
β1
a3
C3
A2
δ1
β3
γ1
A3
C2
B3
δ2
a2
B2
γ2
β2

Figure 5

-which will <u>not</u> any more be a shifting of boundary as shown in figure (2).

The idea is shown, for two different cases, in pictures (6) and (7).

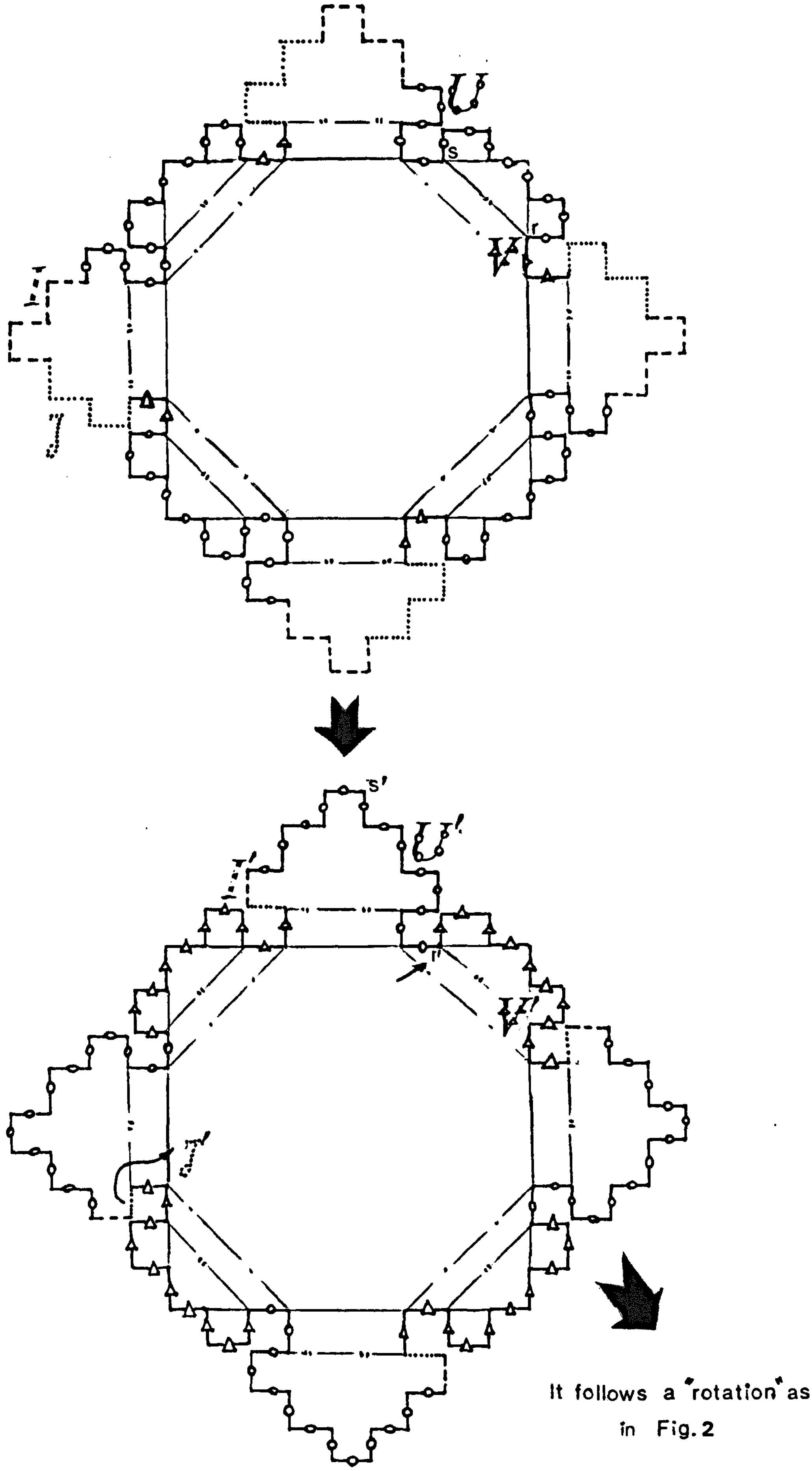
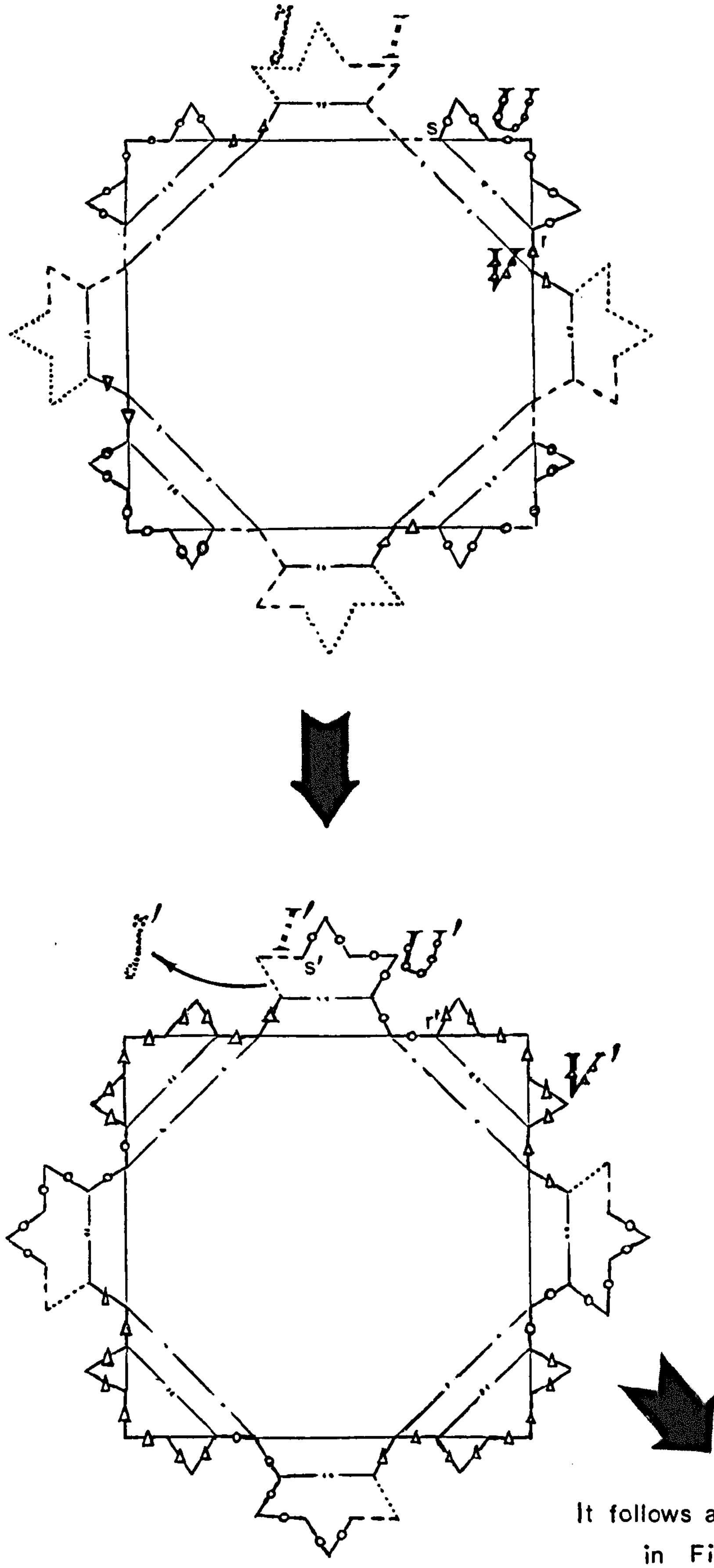


Figure 6



It follows a "rotation" as in Fig. 2

Figure 7

Chapter 4

The purpose of this section is to prove the Rellich theorem for a certain class of functions supported in a tail T, namely: functions which are locally ruled. We will describe this class later on.

We will construct now a piecewise linear foliation of T, such that the leaves of the foliation will be paths having initial points on the segment $s_0 = \partial T \cap \partial \Omega^{\circ}$, and end points on $\partial\Omega$.

We are going to construct these paths in stages, and at each stage the corresponding portions of the paths will be line segments.

We will consider T as a rampant union of self similar regions of decreasing size :

$$T = A_1 + \sum_{p=0}^{\infty} \sum_{i=1}^{N_0 N^p} A_i^{p+2}$$

where each A_i^p is a $cc(\Omega^p - \Omega^{p-1})$ contained in the tail T, and where $N_0 \in \mathbb{N}$, $N_0 < N$.

As shown in figure (1), for the Koch snowflake we have : $N_0 = 2$, and the A_i^p are triangles of side $1/3^p$.

Let us do first the foliation for the case of the Koch snowflake.

We start the paths at points in s_0 and we first foliate A_1, by mapping, linearly, the segment s_0 in the two other segments of ∂A_1, in the way shown in figure (2), and we now join points of s_0

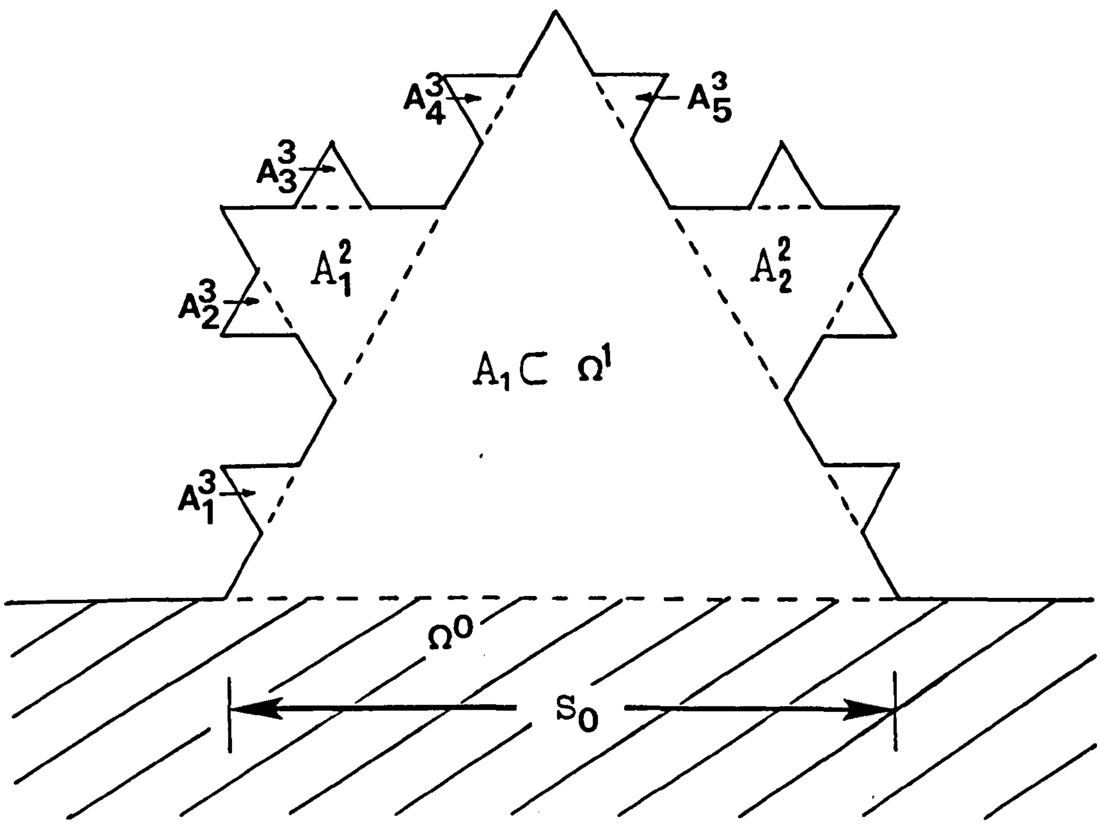


Figure 1

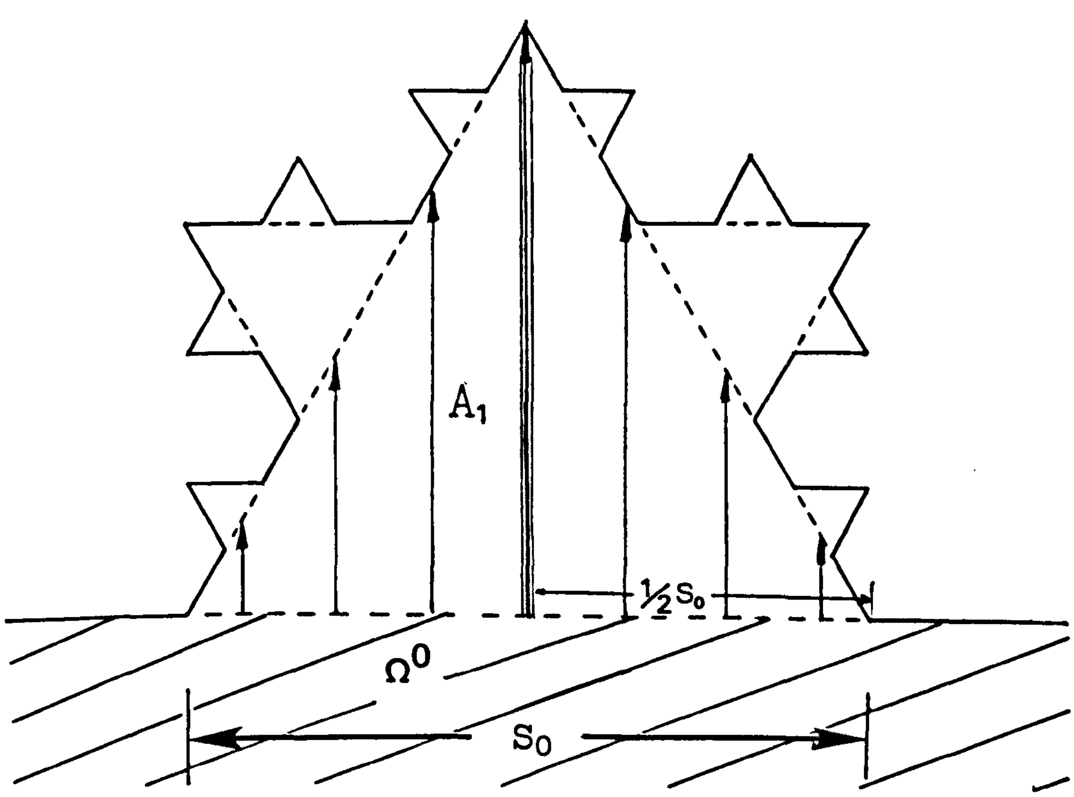


Figure 2

with their images in the other two segments of A_1 .

These segments are the first stage in the construction of the paths (see figure (3)).

The second stage of the construction of the piecewise linear paths is as follows :

We will foliate now the triangles A_i^2 , $i \in \{1,2\}$, in much the same way in which we foliated A_1 .

The starting points of the new stage of the paths foliating A_i^2 will be the points of the segment $s_i^1 = \partial A_i^2 \cap \partial A_1$.

We map, linearly, the segments s_i^1 , $i \in \{1,2\}$ in the other two segments of ∂A_i^2 , and again join points of s_i^1 with their images in the other two segments of A_i^2 , as shown in figure (4) .

Notice that A_i^3 , $i \in \{1,4,5,8\}$ have boundary in common with A_1 (see figure (4)). Hence, we can foliate these A_i^3 in the same way as we did the A_i^2 .

The iteration of this process foliating A_i^3 ; $i \in \{2,3,6,7\}$ from starting points in the segments $s_{i,k}^2 = \partial A_k^2 \cap \partial A_i^3$;

$i \in \{2,3,6,7\}$; $k \in \{1,2\}$ will complete the third stage in the construction of the paths.

This process, iterated *ad infinitum* , will give us the piecewise linear foliation that we set out to construct.

Let us consider now a triangle A_i^p , and let A_j^{p-1} be the unique A_k^q , $q \le p-1$; $k \in \{1,2,\dots,N_0 N^q\}$; with boundary in common with A_i^p ; and let us consider the segment

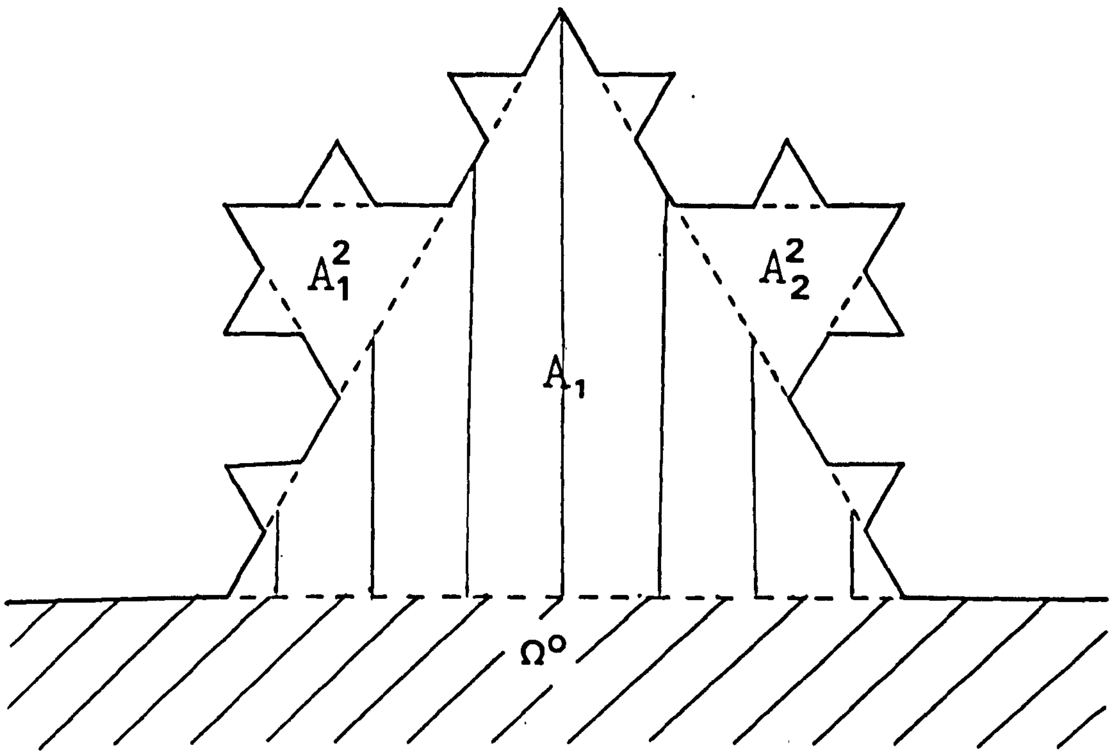


Figure 3

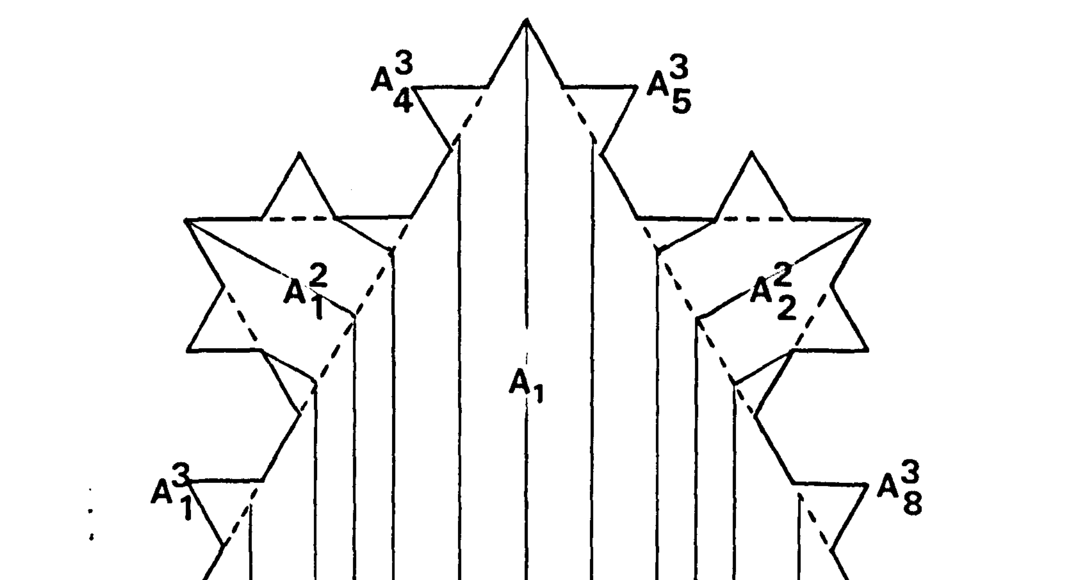


Figure 4

$s_i^{p-1} = \partial A_i^p \cap A_j^{p-1}$.

Therefore, as shown in figure (5) , taking axes in the directions of s_i^{p-1} and the direction of the segments contained in the parts given in the foliation, we obtain a system of local cartesian coordinates.

Let us now construct an analogous piecewise linear foliation for the case in which

$$T = A_1 + \sum_{p=0}^{N_0 N^p} A_i^{p+2}$$

where the self similar A_i^p are convex polygons.

Let us choose a certain convex shape, e.g. a pentagon, and show how the foliation is constructed. The general case will follow immediately.

Since the process that yields the foliation is identical for every A_i^p , we will only show how it is done for A_1 .

Let V_i ; $i \in \{1, 2, 3, 4, 5\}$ be the set of vertices of our A_1 ; the segment $s_0 = \partial A_1 \cap \partial\Omega^0$ is the segment

$$\overline{V_1 V_5} \;,$$

as shown in figure (6).

The first stage of the construction of the piecewise linear paths is the foliation of the triangle $V_1 V_2 V_5$.

We will proceed in a way analogous to the one used to foliate

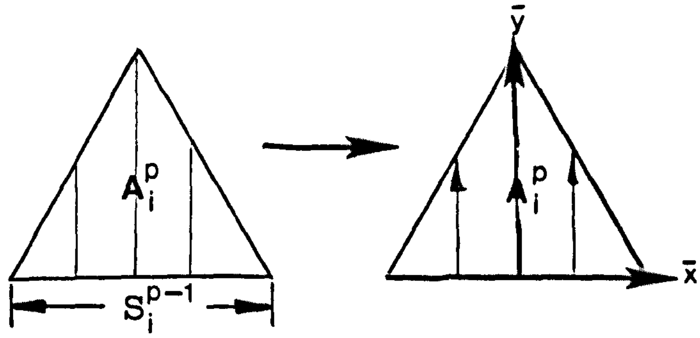


Figure 5

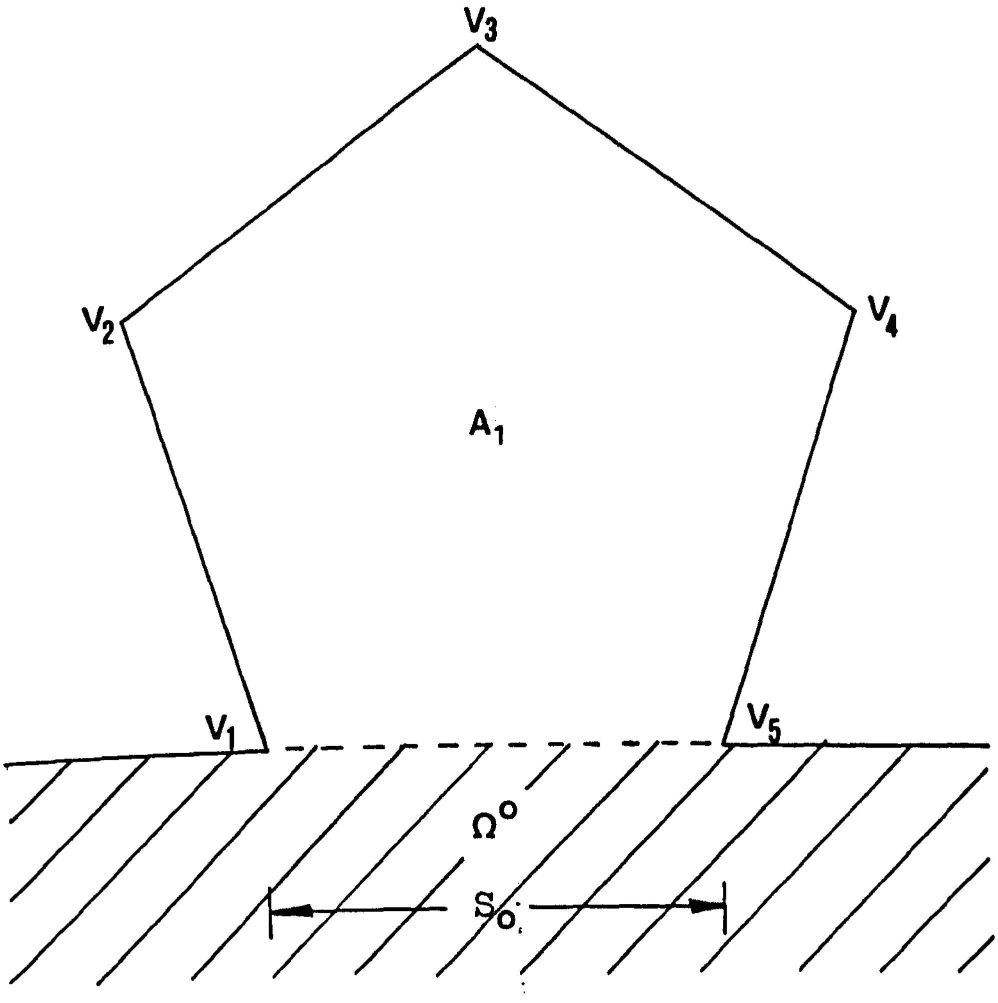


Figure 6

the triangle A_1 in the case of the snowflake.

Let P_1 be the middle point of the segment s_0 and consider the segment

$$\overline{V_2 P_1} \ .$$

As before, we map s_0 into the segments

$$\overline{V_1 V_2} \quad \text{and} \quad \overline{V_2 V_5}$$

in a linear way, and such that :

$$\overline{V_1 P_1} \quad \text{is mapped into} \quad \overline{V_1 V_2} \ , \ \text{and}$$
$$\overline{P_1 V_5} \quad \text{is mapped into} \quad \overline{V_2 V_5} \ ,$$

as shown in figure (7) .

As before, we now join points of s_0 with their corresponding images in

$$\overline{V_1 V_2} \quad \text{and} \quad \overline{V_2 V_5} \ ,$$

by line segments, these segments are the first stage in the construction of the paths (see figure (8)) .

These last segments are parallel to the segment

$$\overline{V_2 P_1} \ .$$

The second stage of the construction of the piecewise linear paths, is the foliation of the triangle $V_2V_3V_5$ in much the same way :

Let P_2 be the middle point of

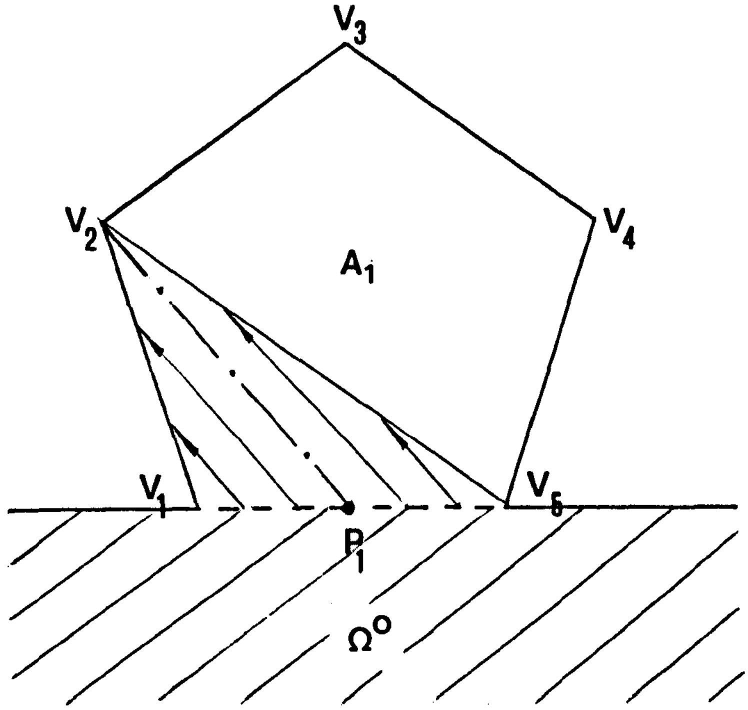


Figure 7

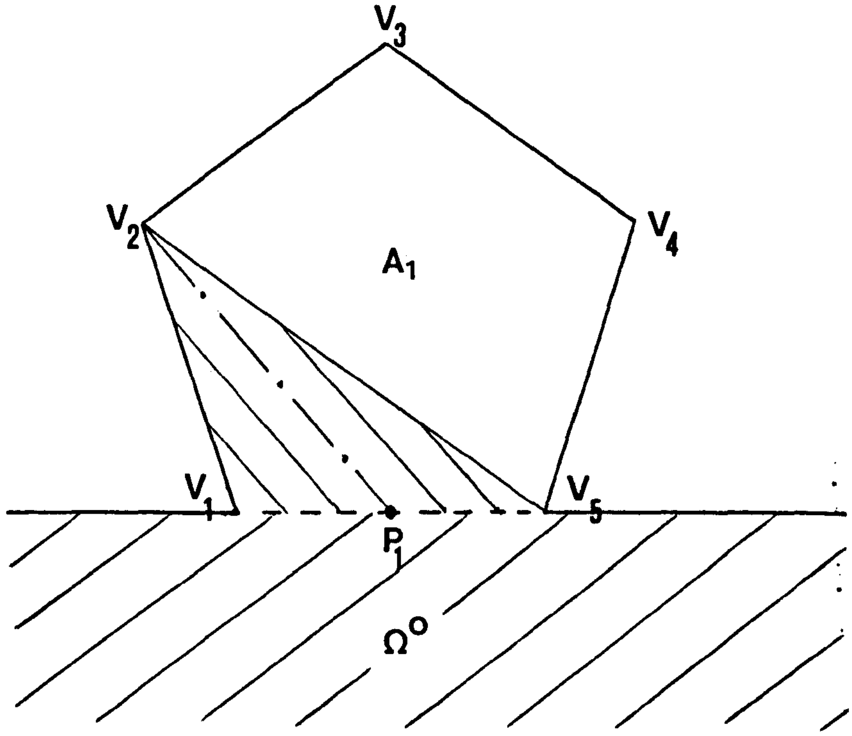


Figure 8

$$\overline{V_2 V_5} \ .$$

The role played by the segment s_0 in the first stage of the construction of the foliation, will be played now by the segment

$$\overline{V_2 V_5} \ ,$$

just as the role played by the point P_1 will be now played by the point P_2 .

As before, we consider the segment

$\overline{V_3 P_2}$, and we map $\overline{V_2 V_5}$, linearly, into $\overline{V_2 V_3}$ and $\overline{V_3 V_5}$,

in such a way that the image of

$\overline{V_2 P_2}$ is $\overline{V_2 V_3}$, and the image of $\overline{P_2 V_5}$ is $\overline{V_3 V_5}$ (see figure (9) (a)).

We now join points of

$\overline{V_2 V_5}$ with their images in $\overline{V_2 V_3}$ and $\overline{V_3 V_5}$

by line segments (all of them parallel to the segment

$\overline{V_3 P_2}$), and these segments are the second stage of the construction of the paths (see figure (9) (b)).

It remains to iterate the process, once more, in the triangle $V_3V_4V_5$, with starting points in the segment

$\overline{V_3 V_5}$, which has P_3 as middle point.

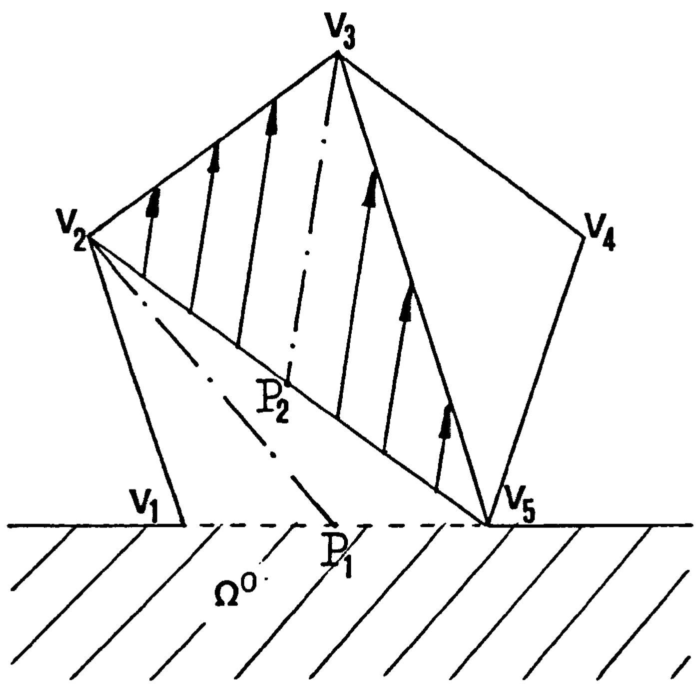


Figure 9(a)

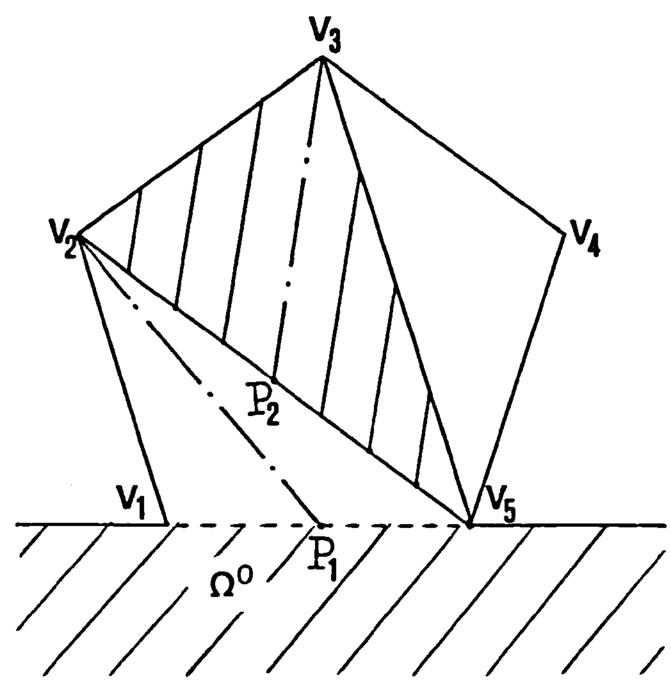


Figure 9(b)

We can see that this process can be applied to the construction of a piecewise linear foliation in any given convex polygon A_1 .

Note 1 : The local cartesian system of coordinates depicted before for A_i^p in the case of the Koch snowflake, has been replaced, in the case of the pentagon, by three systems of local coordinates, as shown in figure (10) ; and, in the general case, if ∂A_i^p is a polygon of k vertices, we obtain $k - 2 = c(\Omega)$ new systems of local coordinates.

In order to regain the orthogonality of the cartesian systems depicted before for A_i^p in the case of the Koch snowflake, we need a maximum of $k-2=c(\Omega)$ linear mappings.

Hence, the case of a convex polygon of k vertices is not fundamentally different from the case of the Koch snowflake: in fact, we can select a foliation having no paths containing a segment in A_1 parallel to $\partial A_1 \cap \partial \Omega^0$. We will refer to this property of our foliation by the expresion quasi orthogonality of our paths.

This quasi orthogonality of the leaves of our piecewise linear foliation is an important property used in the reasonings that follow, and we will restrict ourselves to foliations that have it.

Note 2 : Notice that we can construct a foliation having quasi orthogonality of its leaves in more than one way. For instance, in the case in which A_1 is a pentagon, we can subdivide A_1 into three different triangles, e.g. $V_1V_3V_5$, $V_1V_2V_3$ and $V_3V_4V_5$, as shown in figure (11) (a) , and we start by foliating $V_1V_3V_5$ in a cartesian way, as shown in figure (11) (b) ;

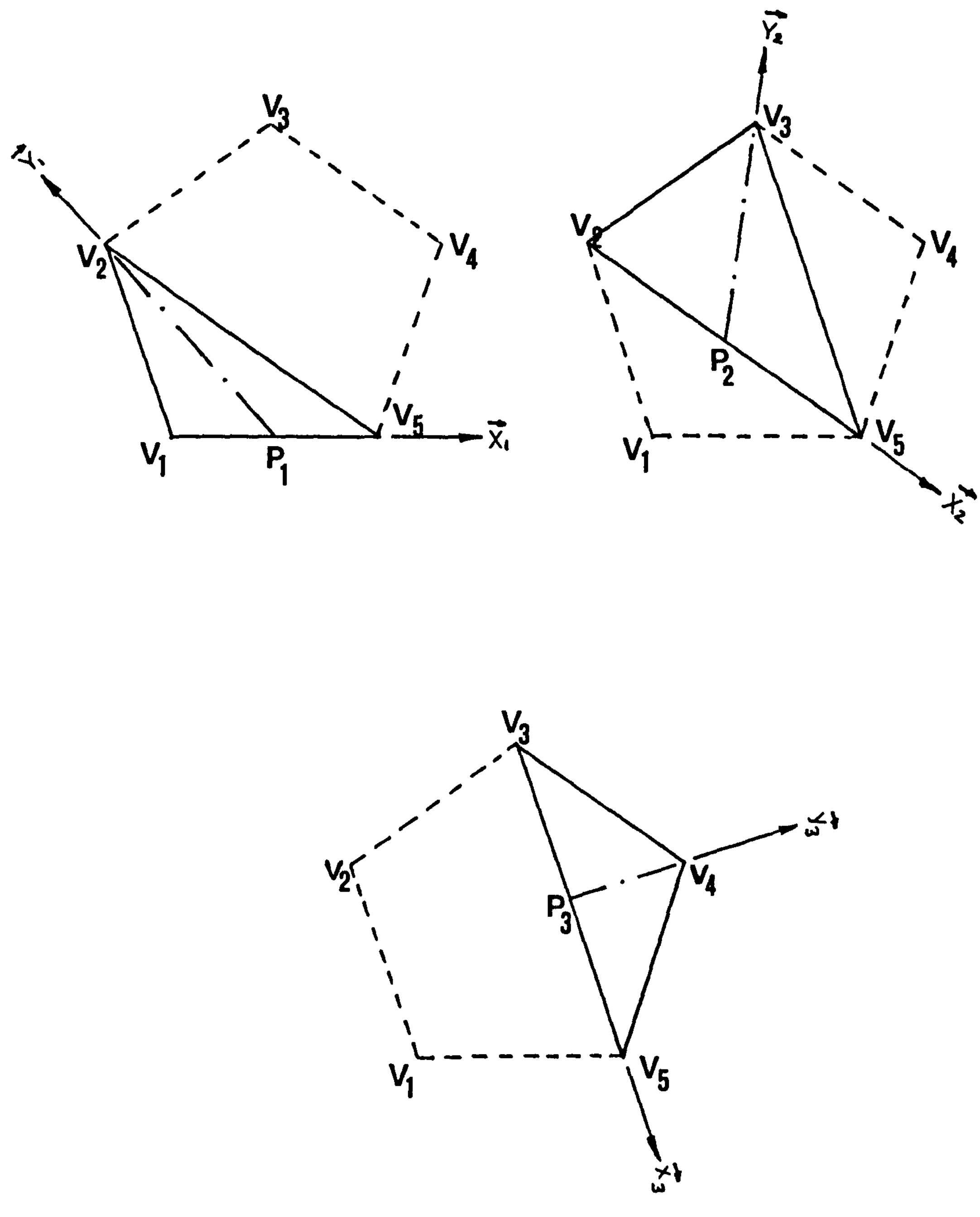


Figure 10

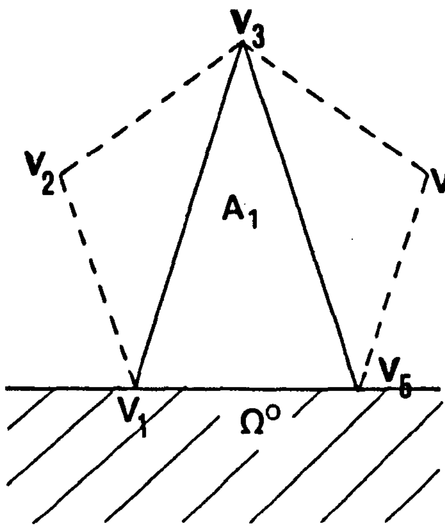


Figure 11(a)

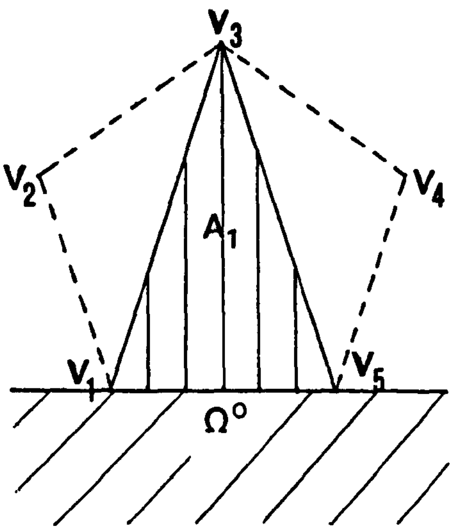


Figure 11(b)

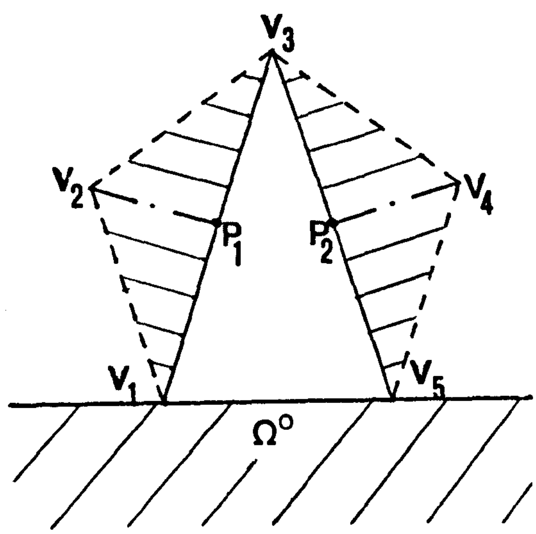


Figure 11(c)

then, with P_1 as the middle point of $\overline{V_1 V_3}$, and

with P_2 as the middle point of $\overline{V_3 V_5}$,

we draw the segments

$$\overline{V_2 P_1} \text{ and } \overline{V_4 P_2}$$

(see figure (11) (c)) and proceed to foliate the remaining triangles $V_1V_2V_3$ and $V_3V_4V_5$ with leaves that will be segments parallel to

$$\overline{V_2 P_1} \text{ and } \overline{V_4 P_2}$$

respectively, as shown in figure (11) (c) .

Notice that we need to introduce <u>only one</u> extra linear mapping, in order to reduce our new three systems of local coordinates to local orthogonality.

Notice also that we could have subdivided A_1 in any other finite number of subregions, not necessarily triangles, and carried on a similar process.

Let us now extend this process to the case in which A_1 is the non rampant sum of convex polygons. Again, the treatment of one example will suffice.

Let us take, for instance, the case in which A_1 is a room-and-passage set : the non rampant sum of a square room and a narrow passage ; and let us denote the room by R and the passage by P (see figure (12) (a)). We start the process by foliating P , using either of the two methods described above for the pentagon

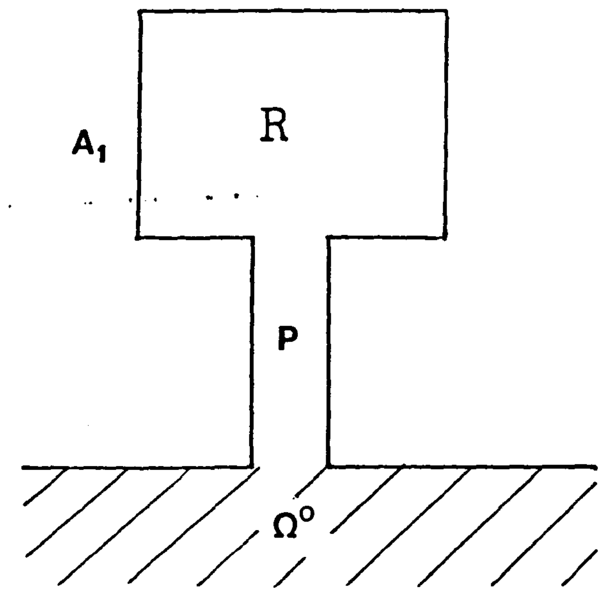


Figure 12(a)

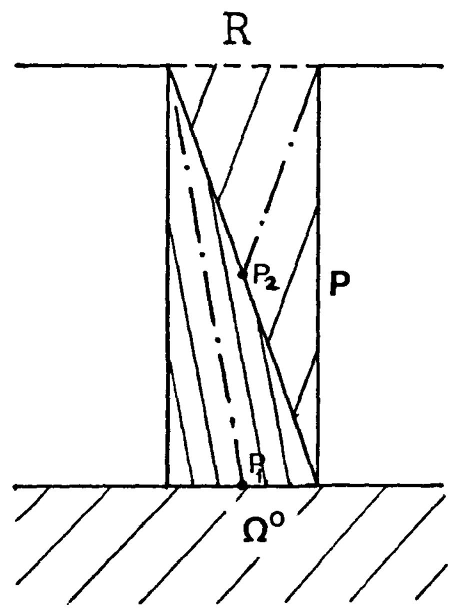


Figure 12(b)

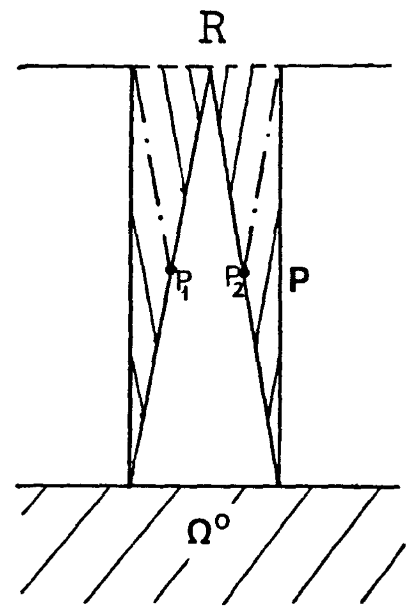


Figure 12(c)

(as shown in figures (12) (b) and (12) (c)).

Next we have to foliate R , which is convex, but now the starting points of the sections of the paths that will foliate R will be in a segment <u>much</u> smaller than the side of the square R .

In order to foliate R , we divide it three regions (as shown in figure (13) (a)) : R_1 , R_2 , R_3 ; R_2 a triangle; we foliate R_2 in the way already described, and it remains to foliate R_1 and R_3 ...which are both convex (see figures (13) (b) and (13) (c)) and the starting points of the new sections of the leaves will be in a <u>whole side</u> of the polygons; hence we are in the former cases already studied.

Notice that the point Q , in figure (13) (a), can have an infinite number of locations in any of the segments

$$\overline{V_1 V_2}\,,\quad \overline{V_2 V_3}\,,\quad \overline{V_3 V_4}\quad :$$

the aim of the subdivision R_1 , R_2 , R_3 was to reduce the foliation of R , starting from

$$\partial R \cap \partial P \subset\subset \overline{V_1 V_4}$$

to the case in which all regions are convex and the segment to which the initial points belong is strictly one side of the polygon ∂R_i ; $i \in \{1, 2, 3\}$.

We emphasize that this can be done in an infinite number of ways, and for the case in which A_t is an arbitrary (finite) non rampant sum of convex sets.

Some properties of the piecewise linear foliation

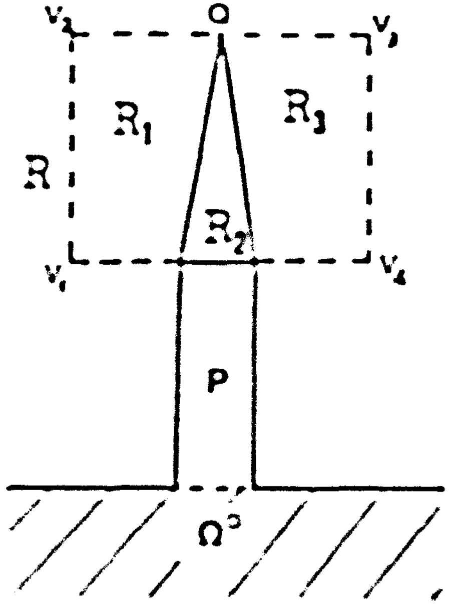


Figure 13(a)

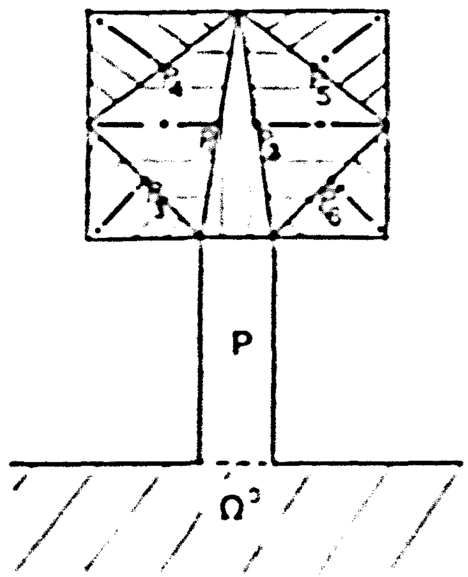


Figure 13(b)

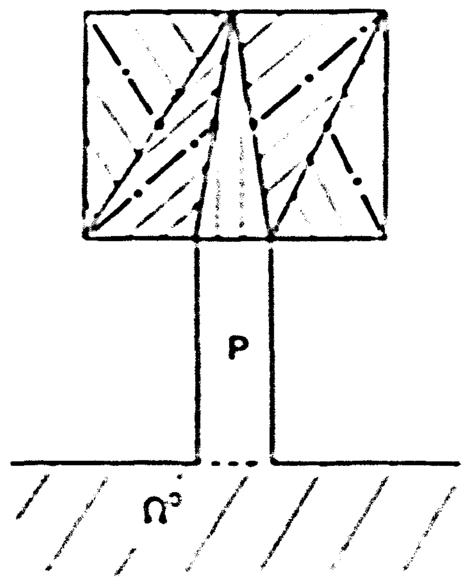


Figure 13(c)

Property 1 :

When calculating the integrals

$$\iint_{\Omega^\varepsilon} u^2 \quad \text{and} \quad \iint_{\Omega} \mathrm{grad}^2 u$$

for a function u supported on a tail T we may write:

$$\iint_{\Omega^\varepsilon} u^2 = \iint_{T^\varepsilon} u^2 \quad \text{and} \quad \iint_{\Omega} \mathrm{grad}^2 u = \iint_{T} \mathrm{grad}^2 u \quad ;$$

and, since the region of integration T is a non rampant sum of regions A_i^p , we can then express the above integrals as sums of integrals over the regions A_i^p and $(A_i^p)^\varepsilon$ for the different values of i and p .

Hence, the property of quasi orthogonality for the paths of the foliation, stated in Note 2 above, will introduce --when performing the $k - 2 = c(\Omega)$ linear transformations indicated in Note 2 -- at most $k - 2 = c(\Omega)$ different constant jacobians in each of the integrals

$$\iint_{(A_i^p)^\varepsilon} u^2 \quad \text{and} \quad \iint_{A_i^p} \mathrm{grad}^2 u \quad .$$

Then, the set of all the jacobians referred to, is bounded above and below. Therefore, for the purposses of our estimation of

$$Q^\varepsilon(\Omega) = \frac{\displaystyle\iint_{\Omega^\varepsilon} u^2}{\displaystyle\iint_{\Omega} \mathrm{grad}^2 u} \quad ,$$

the general case is not fundamentally different from the cartesian case of the Koch snowflake.

Property 2 :

The total length of each path in our foliation is bounded by a constant $c = c(\Omega)$:

Let P be a path from the foliation constructed above; let $l(P)$ be the length of P, and let $l(P)(A_i^p)$ be the length of the path P inside the region A_i^p. Clearly,

$$l(P)(A_i^p) \leq (k-2) \operatorname{diam}(A_i^p)$$

$$= c(\Omega) \operatorname{diam}(A_i^p) = c(\Omega) \operatorname{diam}(A_1)/n^p = c(\Omega)/n^p \quad ;$$

where the integer n is the one in the (n, N) process of replacement.

Given $p \in \mathbb{N}$, there exists at most <u>one</u> index $i = i_p$ such that $P \cap A_{i_p}{}^p \neq \emptyset$.

Hence

$$l(P) = \sum_{p=1}^{\infty} l(P)(A_{i_p}^p) \leq c(\Omega) \sum_{p=1}^{\infty} \frac{1}{n^p}$$

$$= c(\Omega) \frac{1}{n} \frac{1}{1 - \frac{1}{n}} = c(\Omega)$$

Property 3 :

We emphasized that there was an infinity of possible ways of constructing the piecewise linear foliation.

We had obtained a set of at most $k-2$ local systems of coordinates on each A_j^p, given by pairs of axes

$$(\vec{x_i}, \vec{y_i}) \ , \ i \in \mathbb{N} \quad , \quad i \leq k-2 \ .$$

Let us call I the set of these indices; k is always the total number of vertices of the polygon ∂A_j^p .

These local systems were not necessarily orthogonal; let us call α_i the angle

$$\sphericalangle(\vec{x}_i \vec{y}_i)_{i \in I} \quad .$$

Because there is an infinite number of ways of choosing the values of the angles α_i , $i \in I$, we can, therefore, select a foliation with a corresponding set of angles $\{ \alpha_i \}_{i \in I}$ such that, no path in the foliation contains a segment in A_1 parallel to
$\partial A_1 \cap \partial \Omega^o$.

Hence, the same is true for any other A_i^p :
given any A_i^p , let A_j^{p-1} be the only A_k^q , $q \leq p-1$,
$k \in \{ 1, 2, ..., N_0 N^q \}$ with boundary in common with ∂A_i^p ;
then no path in the foliation contains a segment in A_i^p parallel to
$A_i^p \cap \partial A_j^{p-1}$.

We will need this property in the proof of the theorem of Rellich, below. Substantially, it means that we can replace, e.g. in A_1 (hence in every A_j^p), the original systems

$\{(\vec{x}_i \vec{y}_i)\}_{i \in I}$ by the set of systems $\{(\overrightarrow{\partial A_1 \cap \partial \Omega^o}, \vec{y}_i)\}_{i \in I}$;
the new systems have the original axis y_i but a new axis x , the same for every $i \in I$, parallel to the segment $\partial A_1 \cap \partial \Omega^o$.

The area subtended by a segment of length δ :
We need a
Property 4 :

Let P be any path in the piecewise linear foliation. Then, there is a constant $c = c(\Omega)$ such that the length of P in Ω^ε fulfills:

$$l(P \cap \Omega^\varepsilon) \leq c(\Omega)\,\varepsilon \;:$$

Given ε, $0 < \varepsilon < 1$, and n, as usual, the integer in the (n, N) process of replacement, let us choose then $p \in \mathbb{N}$

$$\text{such that } \frac{1}{n^{p+1}} \leq \varepsilon \leq \frac{1}{n^p} \qquad \text{(a)}$$

Given A_i^q, $q \geq p$, $i \in \{1, 2, \ldots, N_0 N^q\}$, there is <u>at most one</u> $i = i_q$ such that $A_{i_q}^q \cap P \neq \emptyset$.

On the other hand, as we saw in property 2, given <u>any</u> A_j^q, we have

$$l(P)(A_j^q) \leq (k-2)\,\mathrm{diam}(A_j^q)$$
$$= c(\Omega)\frac{\mathrm{diam}(A_1)}{n^q} = \frac{c(\Omega)}{n^q}$$

Therefore, if we consider the intersection of P with Ω^ε, we have to take into account P in only those $A_j^q \subset \Omega^\varepsilon$, that is, only the values $q \geq p$.

Therefore,

$$l(P \cap \Omega^\varepsilon) \leq \sum_{q=p}^{\infty} \sup_P l(P)(A_{i_q}^q)$$

$$\leq \sum_{q=p}^{\infty} \frac{c(\Omega)}{n^q} = c(\Omega)\frac{1}{n^p}\,\frac{1}{1-\frac{1}{n}} = c(\Omega)\frac{1}{n^p} = n\,c(\Omega)\frac{1}{n^{p+1}}$$

$$= \alpha(\Omega)\,\frac{1}{n^{p+1}} \leq \alpha(\Omega)\,\varepsilon\ .$$

Now we will introduce the concept of ***area subtended*** .

For simplification, in what is left of this chapter, we will consider the size of Ω normalized so that its tails have at most a diameter 1 .

We keep on considering $0 < \varepsilon \leq 1$ thoughout.

We work always in the same tail T .

Let Q^1 and Q^2 be points in $\partial\Omega$. They are endpoints of two paths P_1 and P_2.

We denote $p = p(P)$ the starting point, in the segment s_0, of the path P; and we denote

$$p_1 = p(P_1) \;;\; p_2 = p(P_2) \;.$$

Let δ be the length of the segment

$$\overline{Q^1 Q^2}$$

$$\text{We denote} \quad \mathbf{P} = \{ P \mid p = p(P) \in (p_1, p_2) \}$$

i.e., the set of paths whose starting point is between p_1 and p_2;

$$\Omega^{\varepsilon,\delta} = \{ x \in \Omega^{\varepsilon} \mid \text{there exists } P \in \mathbf{P} \text{ such that } x \in P \}$$

We call $\Omega^{\varepsilon,\delta}$ <u>the part of Ω^{ε} subtended by the segment</u>

$$\overline{Q^1 Q^2} \;;$$

and the area of the part of Ω^{ε} subtended by the segment Q^1Q^2 will be called the area of Ω^{ε} subtended by this segment.

Lemma :

There exists a constant $c = c(\Omega)$ such that, the area subtended by a segment of length δ is no greater than

$$c(\Omega)\, \delta\, \varepsilon^{2-d} \;.$$

Proof : Let us denote

$$(\partial\Omega)_{\mathbf{P}} = \{ x \in \partial\Omega \mid x \text{ is the endpoint of a path } P \in \mathbf{P} \} \;,$$

the part of $\partial\Omega$ subtended by

$$\overline{Q^1 Q^2} \;;\; \mu^1(\overline{Q^1 Q^2}) = \delta \;.$$

In the same way--due to the regularity of the (n, N) process--in which $\mathrm{diam}(\partial T^p) \le c\, 1/n^p$, for a tail T^p, we have that:

There is a constant $c = c(\Omega)$ such that

$$\operatorname{diam}((\partial\Omega)_{\mathbf{P}}) \leq c(\Omega)\,\delta \quad .$$

Case 1 : Let $\delta > \varepsilon$. Let $p \in \mathbb{N}$ be such that

$$\frac{1}{n^{p+1}} \leq \operatorname{diam}(\partial\Omega)_{\mathbf{P}} \leq \frac{1}{n^{p}} \quad ,$$

so that the area subtended is contained in some tail of size p : T^{p} .

Then, by the first lemma proved in Chapter 1, we have, for the area subtended:

$$\mu^{2}(\Omega^{\varepsilon;\delta}) \leq c(\Omega)\,\varepsilon^{2-d}\,(\frac{1}{n^{p}})^{d} = c(\Omega)\,\varepsilon^{2-d}\,n^{dp}\,(\frac{1}{n^{p+1}})^{d}$$

$$\leq c(\Omega)\,n^{dp}\,\varepsilon^{2-d}\operatorname{diam}((\partial\Omega)_{\mathbf{P}})^{d} \leq c(\Omega)\,\varepsilon^{2-d}\,(c(\Omega)\,\delta)^{d}$$

$$= c(\Omega)\varepsilon^{2-d}\,\delta^{d} \leq c(\Omega)\varepsilon^{2-d}\,\delta \quad ,$$

since $d > 1$ and $0 < \delta \leq 1$.

Case 2 : Let $\delta \leq \varepsilon$. Then

$$\mu^{2}(\Omega^{\varepsilon;\delta}) \leq c(\Omega)\,\delta\,\sup\{\mu^{1}(P\cap\Omega^{\varepsilon})\,|\,P\in\mathbf{P}\} \quad ,$$

but, by property 4 , we have

$$\mu^{1}(P\cap\Omega^{\varepsilon}) \leq c(\Omega)\,\varepsilon \quad , \text{ therefore}$$

$$\mu^{2}(\Omega^{\varepsilon;\delta}) \leq c(\Omega)\,\delta\,c(\Omega)\,\varepsilon = c(\Omega)\,\delta\varepsilon \leq c(\Omega)\,\varepsilon^{2-d}\,\delta \quad ,$$

since $0 < \varepsilon \leq 1$ and $0 < 2 - d < 1$.

Note :

The same result will hold if Q^{1} and Q^{2} are points in $\partial\Omega^{j}$, $j \in \mathbb{N}$, instead of being points in $\partial\Omega$.

Local piecewise ruled functions :

We have been working using two concepts: piecewise linearity and self similarity.

We will prove the theorem of Rellich for a certain class of functions defined on the basis of these two concepts.

Let us consider A^1 as a non rampant finite sum of parallel strips, two sides of each strip are parallel to the segment $s_0 = \partial A^1 \cap \partial \Omega^0$ (see figure (14) for the case of the Koch snowflake).

We define a likewise finite sum for each A_i^p, by self-similarity.

A local piecewise ruled function u, $\sup u \subseteq T$, is a function ruled on each strip of T (with respect to the system (x,y), where the local axis x is parallel to the strips) in the following sense:

Within each strip, we have $u(c_1,y)$ linear in the variable y and $u(x,c_2)$ linear in x.

The widths of the different strips are not necessarily the same.

Theorem : Let u be a local piecewise ruled function. Then $\bar{Q}^\varepsilon(u) \leq c(\Omega)\, \varepsilon^{2-d}$, for $0 < \varepsilon \leq 1$.

Proof : Let u be such a function.

We will foliate the tail T with a foliation allowed by the Property 3: no path in the foliation of A^1 contains a segment parallel to the segment s_0. The analogous statement holds for any A_i^p.

We will introduce a couple of simplifications beforehand:

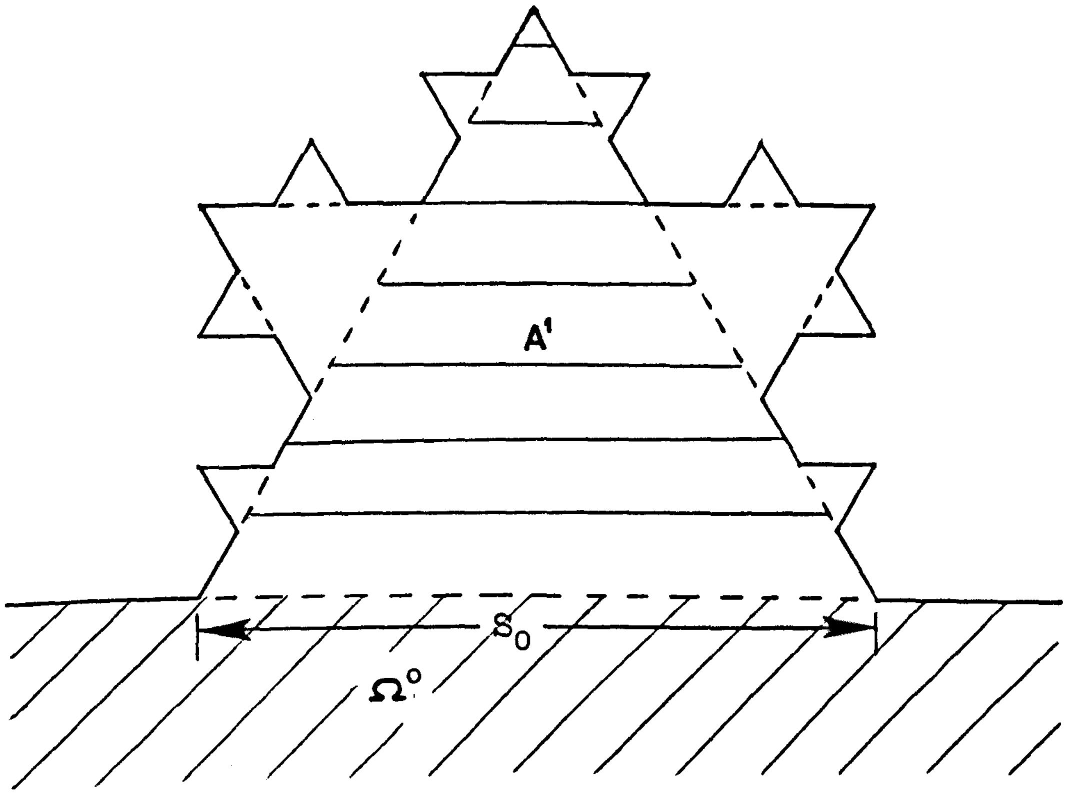


Figure 14

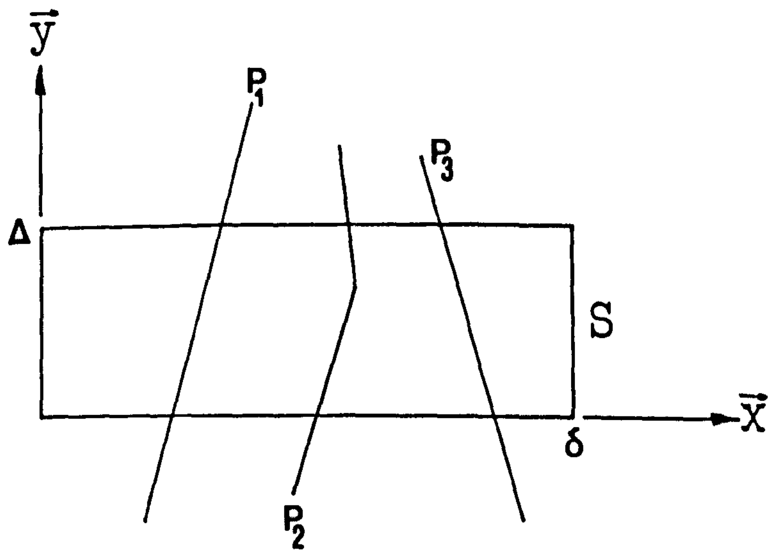


Figure 15

1) By subdividing, if necessary, each strip into sufficiently fine ones, we can work with rectangular strips, with an arbitrarily good approximation to $\overline{Q}^{\varepsilon}(u)$.

2) In what follows, we will work with the integrals

$$\int_{P\cap S} \mathrm{grad}^2\, u\, ds \quad ,$$

in which $P\cap S$ is the path P intersected with the strip S ,and ds is the differential of the arc of curve corresponding to the path P . Let us denote by Δ and δ the width and length of S as seen in figure (15); and let us consider a local cartesian system of coordinates

$$(\vec{x},\vec{y}) \quad \text{, with } \vec{x}$$

parallel to the segment s_0 in the case in which $S \subset A^1$, and the analogous condition for the case $S \subset A_1^p$ (see figure (15)).

Then our foliation, due to property 3 , allows us to write

$$\int_{P\cap S} \mathrm{grad}^2\, u\, ds \sim \int_0^{\Delta} \mathrm{grad}^2\, u\, dy \quad .$$

Now for the proof. Let $x \in T$. There exists only one path P in the foliation, such that $x \in P$. For that P , we denote by $s(x)$ the arc length corresponding to the point $x \in P$ in the path P ; remembering that the starting point of P is $p = p(P)$ in s_0 , and therefore $s(p) = s(p(P)) = 0$.

We can then write:

$$u(x) = \int_0^{s(x)} \frac{du}{ds}\, ds = \int_0^{s(x)} (u_x x_s + u_y y_s)\, ds$$

$$\leq \int_0^{s(x)} \sqrt[2]{\mathrm{grad}^2\, u}\; ds \leq \sqrt[2]{\int_0^{s(x)} \mathrm{grad}^2\, u\, ds}\; \sqrt[2]{l(P)}$$

We took the fact that $\mathrm{supp}\, u \subset T$ into consideration, $l(P)$ is the length of the path P. By property 2 we have

$l(P) \leq c(\Omega)$; therefore

$$u(x) \leq \alpha(\Omega) \sqrt[2]{\int_0^{s(x)} \mathrm{grad}^2\, u\, ds} \quad \text{, and}$$

$$u^2(x) \leq \alpha(\Omega) \int_0^{s(x)} \mathrm{grad}^2\, u\, ds$$

Now we enumerate the strips as $S_j^{p,i}$; the double index p,i indicates that the strip is in A_i^p ; the subindex j enumerates the different strips in A_i^p .

We denote by $P_j^{p,i}$, the intersection of the path P and the strip $S_j^{p,i}$. Then

$$\iint_{\Omega^\varepsilon} u^2 \leq \iint_{\Omega^\varepsilon} \Big[\int_0^{s(x)} \mathrm{grad}^2\, u\, ds \Big]\, dx$$

$$\text{and} \iint_{\Omega^\varepsilon} \Big[\int_0^{s(x)} \mathrm{grad}^2\, u\, ds \Big] dx = \sum_{q,l,m} \iint_{\Omega^\varepsilon \cap S_m^{q,l}} \Big[\int_0^{s(x)} \mathrm{grad}^2\, u\, ds \Big] dx \ .$$

Let $x \in \Omega^\varepsilon \cap S_m^{q,l}$, and consider $\bar{x} \in \partial S_m^{q,l}$, such that

$$\int_0^{s(x)} \operatorname{grad}^2 u \, ds \le \int_0^{s(\bar{x})} \operatorname{grad}^2 u \, ds \ .$$

We may then write

$$\iint_{\Omega^\varepsilon} \Big[\int_0^{s(x)} \operatorname{grad}^2 u \, ds \Big] dx \le \sum_{q,l,m} \iint_{\Omega^\varepsilon \cap S_m^{q,l}} \Big[\int_0^{s(\bar{x})} \operatorname{grad}^2 u \, ds \Big] dx$$

$$= \sum_{q,l,m} \iint_{\Omega^\varepsilon \cap S_m^{q,l}} \Big[\sum_{p,i,j} \int_{P_j^{p,i}} \operatorname{grad}^2 u \, ds \Big] dx$$

$$\sim \sum_{q,l,m} \iint_{\Omega^\varepsilon \cap S_m^{q,l}} \Big[\sum_{p,i,j} \int_0^{\Delta_j^{p,i}} \operatorname{grad}^2 u \, dy \Big] dx$$

We used our second simplification, stated above in the proof.

Then

$$\iint_{\Omega^\varepsilon} \{ \int_0^{s(x)} \operatorname{grad}^2 u \, ds \} \, dx$$

$$\le \alpha(\Omega) \sum_{q,l,m} \iint_{\Omega^\varepsilon \cap S_m^{q,l}} \{ \sum_{p,i,j} \sup_{\substack{P \\ P \cap S_j^{p,i} \neq \emptyset}} \int_0^{\Delta_j^{p,i}} \operatorname{grad}^2 u \, dy \} \, dx$$

$$= \alpha(\Omega) \sum_{q,l,m} \iint_{\Omega^{\varepsilon} \cap S_m^{q,l}} \{ \sum_{p,i,j} I_j^{p,i} \} \, dx \ ,$$

where now $I_j^{p,i}$ depends solely on the strip $S_j^{p,i}$.

Therefore:

$$\iint_{\Omega^{\varepsilon}} \{ \int_0^{s(x)} \operatorname{grad}^2 u \, ds \} \, dx \leq \alpha(\Omega) \sum_{q,l,m} \sum_{p,i,j} I_j^{p,i} \iint_{\Omega^{\varepsilon} \cap S_m^{q,l}} dx$$

$$= \sum_{q,l,m} \sum_{p,i,j} I_j^{p,i} \, \mu^2(\Omega^{\varepsilon} \cap S_m^{q,l}) \ ;$$

now we have to remember that the indices p , i , j , correspond to all $S_j^{p,i}$ for which a path P intersects in order to reach x (and $\bar{x}$) in $S_m^{q,l}$; therefore, when inverting the order of

$$\sum_{q,l,m} \text{ and } \sum_{p,i,j} \ ,$$

then, $I_j^{p,i}$ will be <u>common factor of all values</u>

$$\mu^2 \, (\Omega^{\varepsilon} \cap S_m^{q,l})$$

corresponding to the points in $S_m^{q,l} \cap \Omega^{\varepsilon}$ that can be reached by a path intersecting $S_j^{p,i}$.

But the sum of all these values is the area subtended by a segment of length $\delta_j^{p,i}$. Applying our former lemma(*) we obtain

$$\iint_{\Omega^{\varepsilon}} \{ \int_0^{s(x)} \operatorname{grad}^2 u \, ds \} \, dx \leq \alpha(\Omega) \sum_{p,i,j} I_j^{p,i} \, \alpha(\Omega) \, \varepsilon^{2-d} \, \delta_j^{p,i}$$

(*) in page 77

$$= c(\Omega)\, \varepsilon^{2-d} \sum_{p,i,j} l_j^{p,i}\ \delta_j^{p,i} \ .$$

Since

$$\iint_\Omega \operatorname{grad}^2 u = \sum_{p,i,j} \iint_{S_j^{p,i}} \operatorname{grad}^2 u \quad ,$$

then it remains to prove :

$$l_j^{p,i}\ \delta_j^{p,i} \le c \iint_{S_j^{p,i}} \operatorname{grad}^2 u \, dx\, dy \quad ,$$

and here we can concentrate on <u>just one</u> strip $S_j^{p,i}$ and simply drop all indices.

Figure (16) depicts the situation in a strip S of width Δ and length δ ; $u(x,y)$ is depicted in local coordinates on S . In the four corner points of S , u takes the values a , b , α , β . Since we deal with the gradient of u ,we may as well take $a = 0$.

Observing figure (16) , and, since u is ruled, we can write:

$$\frac{h^i}{y} = \frac{b}{\Delta} \quad ; \quad \frac{\alpha}{\delta} = \frac{l^a}{x} \quad ; \text{ and}$$

$$\frac{h^d - \alpha}{y} = \frac{\beta - \alpha}{\Delta} \quad ; \quad \frac{l^t - b}{x} = \frac{\beta - b}{\delta} \quad ;$$

therefore, for our point $\mathfrak{X} = (x,y)$ in the local coordinates, we have:

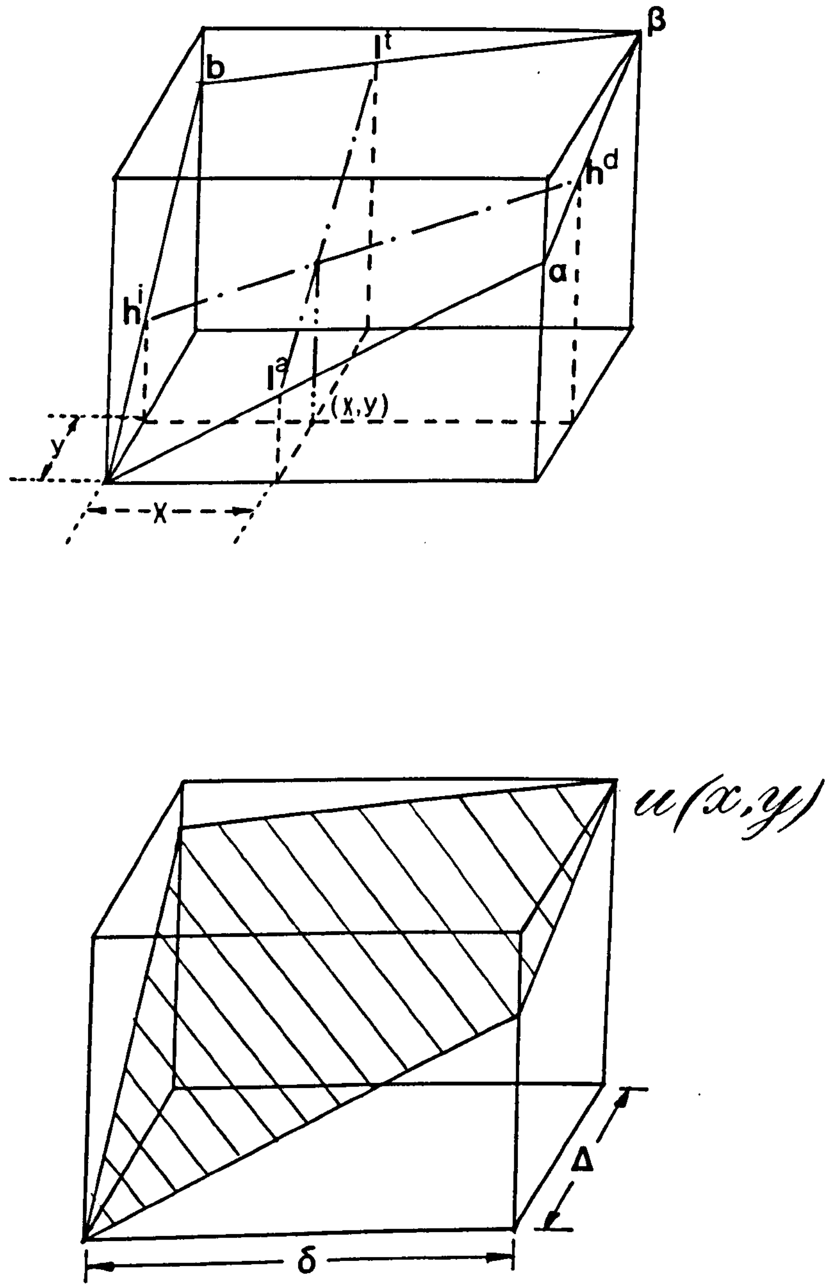


Figure 16

$$u_x = \frac{h^d - h^i}{\delta} = \frac{\frac{\beta - \alpha}{\Delta} y + \alpha - \frac{b}{\Delta} y}{\delta}$$

$$= \frac{y}{\Delta\delta}(\beta - \alpha - b) + \frac{\alpha}{\delta} = \frac{y}{\Delta\delta} p + \frac{\alpha}{\delta} \quad ;$$

Likewise:

$$u_y = \frac{l^t - l^a}{\Delta} = \frac{\frac{\beta - b}{\delta} x + b - x\frac{\alpha}{\delta}}{\Delta}$$

$$= \frac{x}{\Delta\delta}(\beta - b - \alpha) + \frac{b}{\Delta} = \frac{x}{\Delta\delta} p + \frac{b}{\Delta}$$

where $p = \beta - b - \alpha$ is a constant depending only on the strip S .

That gives: $\quad \mathrm{grad}^2 u = u_x{}^2 + u_y{}^2 =$

$$\left\{ \frac{y^2 p^2}{\delta^2 \Delta^2} + 2\frac{y p \alpha}{\delta^2 \Delta} + \frac{\alpha^2}{\delta^2} \right\} + \left\{ \frac{x^2 p^2}{\delta^2 \Delta^2} + 2\frac{x p b}{\delta \Delta^2} + \frac{b^2}{\Delta^2} \right\} \quad .$$

Therefore

$$I\,\delta = \delta \sup_{\substack{P \\ P \cap S \neq \emptyset}} \left\{ \left(\frac{\Delta^3}{3} \frac{p^2}{\delta^2 \Delta^2} + 2\frac{\Delta^2}{2} \frac{p\alpha}{\delta^2 \Delta} + \frac{\alpha^2 \Delta}{\delta^2} \right) + \right.$$

$$\left. \Delta \left(\frac{x^2 p^2}{\delta^2 \Delta^2} + 2x\frac{pb}{\delta \Delta^2} + \frac{b^2}{\Delta^2} \right) \right\}$$

$$\leq \delta \left\{ \frac{\Delta}{\delta^2} \left(\frac{p^2}{3} + p\alpha + \alpha^2 \right) + \frac{1}{\Delta} \left(\frac{\delta^2 p^2}{\delta^2} + 2\frac{\delta |p| b}{\delta} + b^2 \right) \right\}$$

using $0 \leq x \leq \delta$.

Therefore

$$I\,\delta \le \frac{\Delta}{\delta}\left(\frac{p^2}{3} + p\,\alpha + \alpha^2\right) + \frac{\delta}{\Delta}\left(p^2 + 2\,|p|\,b + b^2\right) \qquad (1)$$

On the other hand

$$\iint_S \operatorname{grad}^2 u = \int_0^\delta dx \int_0^\Delta \operatorname{grad}^2 u \, dy$$

$$= \int_0^\delta dx \left\{ \frac{\Delta}{\delta^2}\left(\frac{p^2}{3} + p\alpha + \alpha^2\right) + \frac{1}{\Delta}\left(\frac{x^2 p^2}{\delta^2} + 2\,\frac{pb}{\delta}\,x + b^2\right)\right\}$$

$$= \frac{\Delta}{\delta}\left(\frac{p^2}{3} + p\alpha + \alpha^2\right) + \frac{1}{\Delta}\left(\frac{\delta^3}{3}\,\frac{p^2}{\delta^2} + 2\,\frac{\delta^2}{2}\,\frac{pb}{\delta} + b^2\delta\right)$$

$$= \frac{\Delta}{\delta}\left(\frac{p^2}{3} + p\alpha + \alpha^2\right) + \frac{\delta}{\Delta}\left(\frac{p^2}{3} + pb + b^2\right) \quad ;$$

From (1), and the fact that

$$\frac{p^2}{3} + p\alpha + \alpha^2$$

is a positive definite form in p and α , we will be finished if

$$p^2 + 2\,|p|\,b + b^2 \le c\left(\frac{p^2}{3} + pb + b^2\right) \quad .$$

for some constant c .

If $b = 0$, this is trivial, if $b \neq 0$ it is enough that

$$x^2 \pm 2x + 1 \le \frac{28}{3}\left(x^2 + 3x + 3\right) \quad .$$

Appendix to Chapter 4

Let us consider again the counterexamples of the type rooms-and-passages depicted in the first chapter.

The dimensions of the passages do not mantain the same proportion, as their size diminishes, as we know:

The proportion between width and length tends to zero, as the size of the passages tends to zero.

Consequently,if we proceed to foliate the rooms and passages in the way indicated in chapter 4 , then Property 1 and Property 3 would not hold any longer, since the angles involved would change from passage to passage, as their size diminishes.

Notice that we can, nevertheless, construct a foliation with a total finite number of angles, for all the rooms and passages of every size:

We proceed by foliating the passages, divided into any number of squares, in the way indicated by figure (17); and, once a passage is foliated, we extend the foliation to the lower part of the adyacent room, as shown in figure (18).

Notice that all segments involved in the foliation are at angles 0 , $\pi/4$, and $\pi/2$ with the horizontal.

Once the lower part of the room is foliated, we extend the foliation to the rest of the room, as shown in figure (19); introducing, perhaps, one more angle, if necessary.

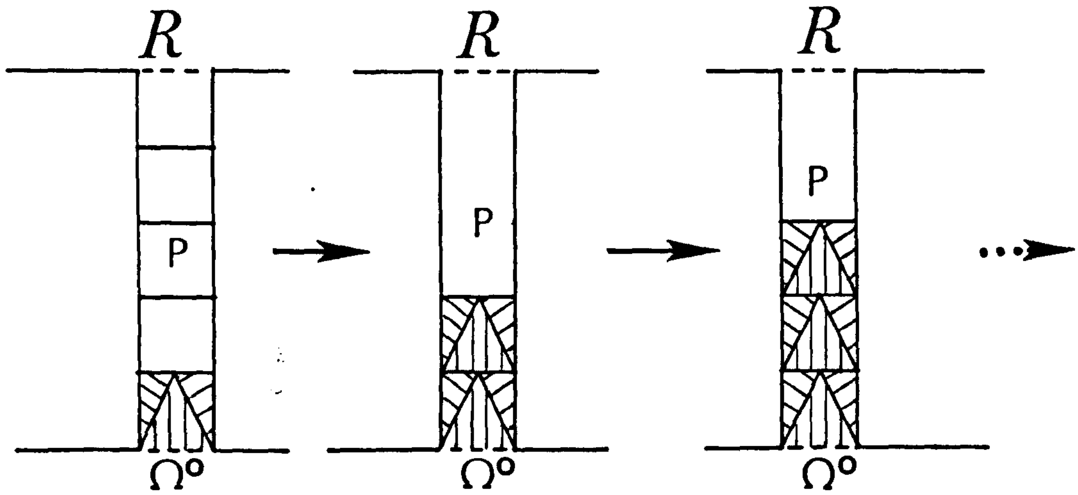


Figure 17

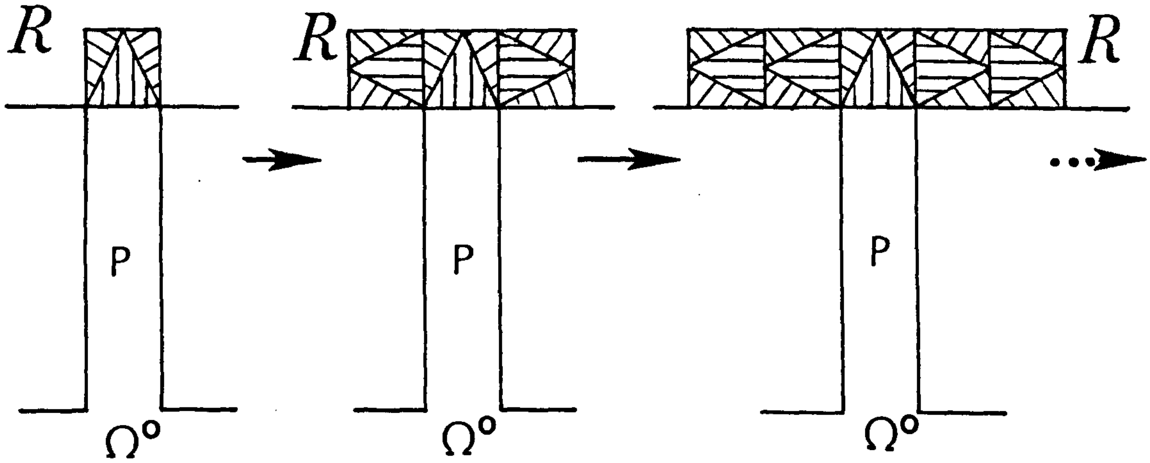


Figure 18

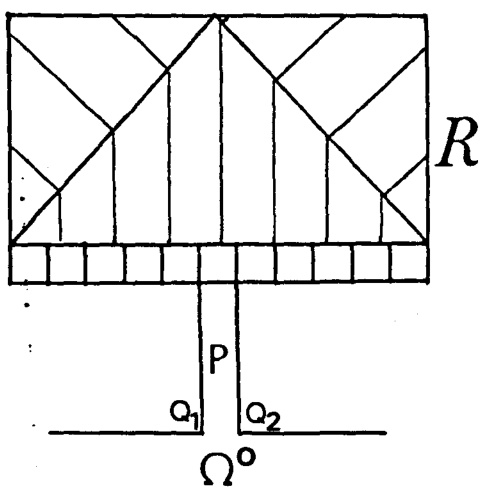


Figure 19

These paths fulfill all of properties 1 , 2 , and 3 .

What they cannot satisfy, however, is the area subtended lemma, which would imply that

$$\frac{\mu^2(\Omega^{\varepsilon,\delta})}{\delta} \to 0 \text{ if } \varepsilon \to 0 .$$

For, let any $\varepsilon \in (0,1)$.

Choose a passage P and an adyacent room R , both contained in Ω^{ε} ; the passage P is joined to the main body Ω° ; let the points Q^1 and Q^2 be points at the intersection of the passage P and Ω° , as shown in figure (19) .

Therefore

$$\mu^1(\overline{Q^1 Q^2}) = \delta \quad ;$$

$$\mu^2(\Omega^{\varepsilon,\delta}) = \mu^2(R) + \mu^2(P) \quad ; \text{ and we know that}$$

$$\frac{\mu^2(R) + \mu^2(P)}{\delta} \to \infty$$

if the sizes of different $R \cup P$ tend to zero.

Notice that the same will happen with <u>any other foliation</u> with properties 1, 2, and 3.

Notice, also, the <u>big gap</u> that exists between the value of the quotient

$$\frac{\mu^2(\Omega^{\varepsilon,\delta})}{\delta}$$

for these counterexamples, and for our fractals.

Chapter 5

We will now construct a new set of paths with properties completely different from those of the paths studied in the last chapter. In the case of the Koch snowflake, the paths studied allowed us to define a local set of cartesian coordinates in each triangle A_i^p. This advantage will be lost now. But the new set of paths constructed will have fundamental properties--to be explained below--totally incompatible with local cartesianism.

We continue with our restriction to a single tail of size 1.

We will construct a piecewise linear foliation of the tail T, and the paths which will be the leaves of this foliation, will have initial points on a segment $s_0 \subset \Omega$, and end points on the boundary $\partial\Omega$; with the property that segments of equal length on s_0 are mapped, by pushing along the leaves, into sets of equal d-Hausdorff measure in $\partial\Omega$.

We are going to construct these paths in stages, and, at each stage, the corresponding portion of the paths will be line segments joining points of one segment with points of another, in a piecewise linear way.

We will explain the procedure in the case of a Koch-snowflake. The way of extending the procedure to cases other than the Koch-snowflake will be seen as a natural one.

We will modify a little the shape of the tail T in a non essential way, by only adding a bit of surface to the triangle $A_1 \subset T$, A_1 of side one, as shown in figure (1).

We call T' the modified tail , and A'_1 the correspondingly modified A_1.

The starting points of the paths will belong to the segment $s_0 \subset \Omega$, shown in figure (1)(c), the paths will foliate the modified tail T' , hence the tail T.

We recall that we are considering a region Ω with fractal boundary $\partial\Omega$, as a non rampant union of regions with the same self similar shape, of decreasing size and disjoint interior, e.g. the Koch-snowflake being a union of triangles of decreasing size and disjoint interior, as shown in figure (1)(a).

We begin by foliating first the region A_1' in Ω^o .

We distinguish, in the region A_1' , the subregion

$$(\Omega^o)^{1/n} = (\Omega^o)^{1/3} = \{ (x,y) \in \Omega^o \,/\, \text{dist}((x,y), \partial\Omega^o) < 1/n \}$$

as shown in figure (2).

We will denote it by $\Omega^{o,1/n}$, for short.

The first stage in the construction of the paths is as follows:

We map s_0 into the segments PQ and QR (see figure (2)) of $\partial\Omega^{o,1/n}$ in a linear way, as shown in figure (2)(a).

We now join points of s_0 with their images in $\partial\Omega^{o,1/n}$ by line segments.

These line segments are the first stage in the construction of the paths, as shown in figure (2)(b).

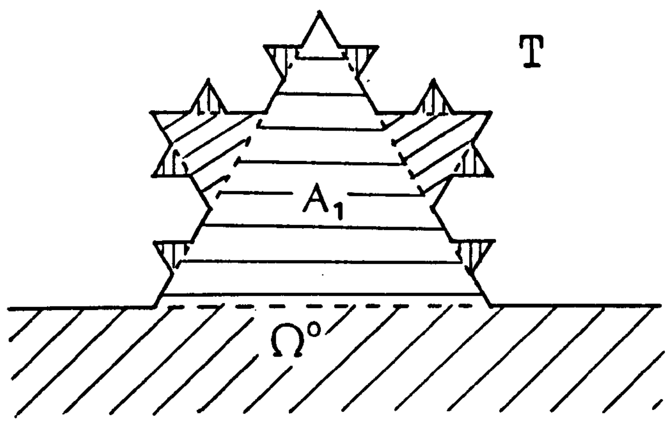


Figure 1(a)

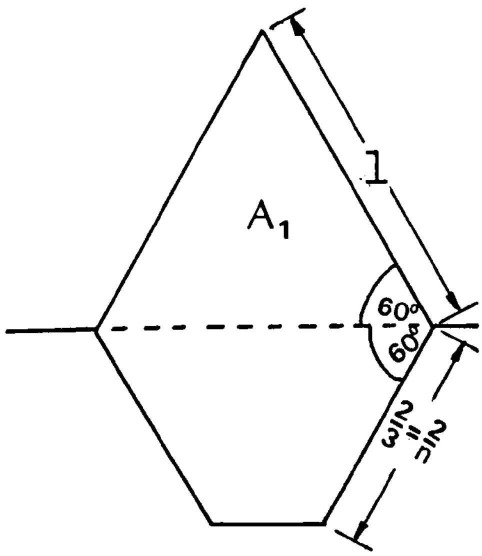


Figure 1(b)

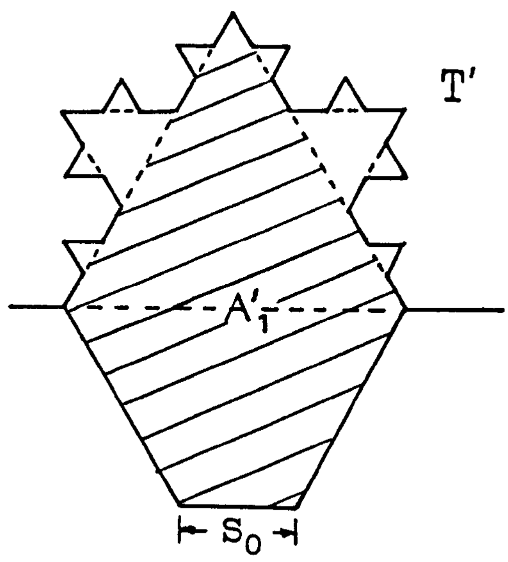


Figure 1(c)

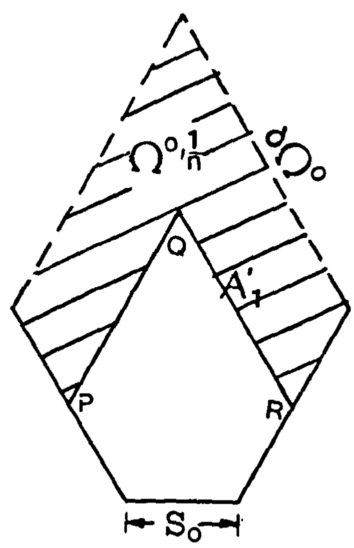


Figure 2

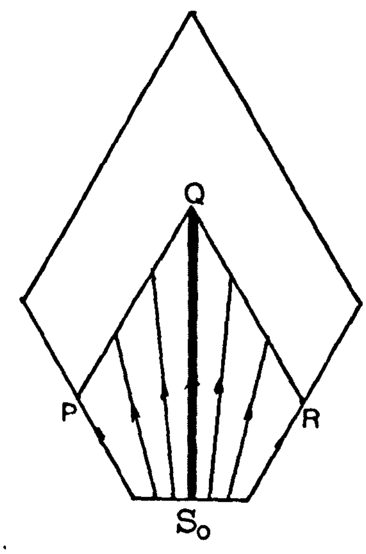


Figure 2(a)

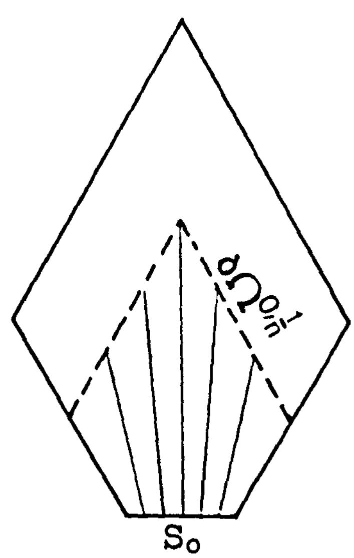


Figure 2(b)

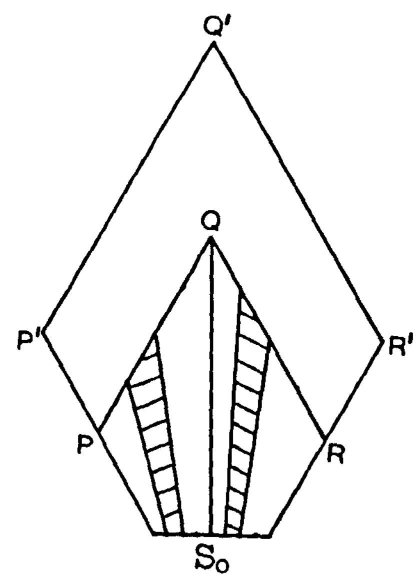


Figure 2(c)

Notice that:

equal segments in s_0 are mapped into equal parts of the other two sides PQ and QR .

<u>Property **J**</u>:

Notice that, taking two segments of equal length s , s' in s_0, then the regions R , R', bounded by the paths originating at end points of s and s', can be transformed one into the other by a transformation of jacobian bounded below and above, and with bounded coefficients in the matrix jacobian, see figure (2)(c).

As we extend the paths, we want to mantain that comparability: we want to mantain property **J** .

We continue with the second stage of the construction of the piecewise linear paths.

In order to ensure that equal parts (in s_0) go to equal parts (in $\partial\Omega$) , we will map the segment PQ onto the section of $\partial\Omega$ delimited by the points P' and Q', as seen in figure (2)(c), and the segment QR onto the section of $\partial\Omega$ delimited by the points Q' and R'.

We associate, therefore, the segment PQ with the segment P'Q' , and, similarly, QR with Q'R'; and, in doing so, we take into consideration for the first time, the following approximation Ω^1, as shown in figure (3).

We divide the segment PQ (also the segment QR) into 4

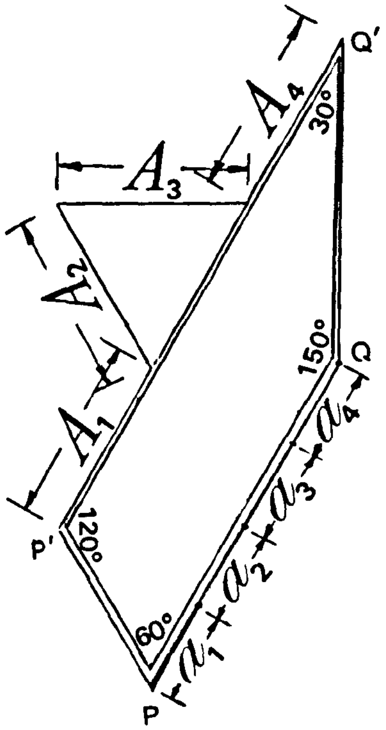


Figure 3

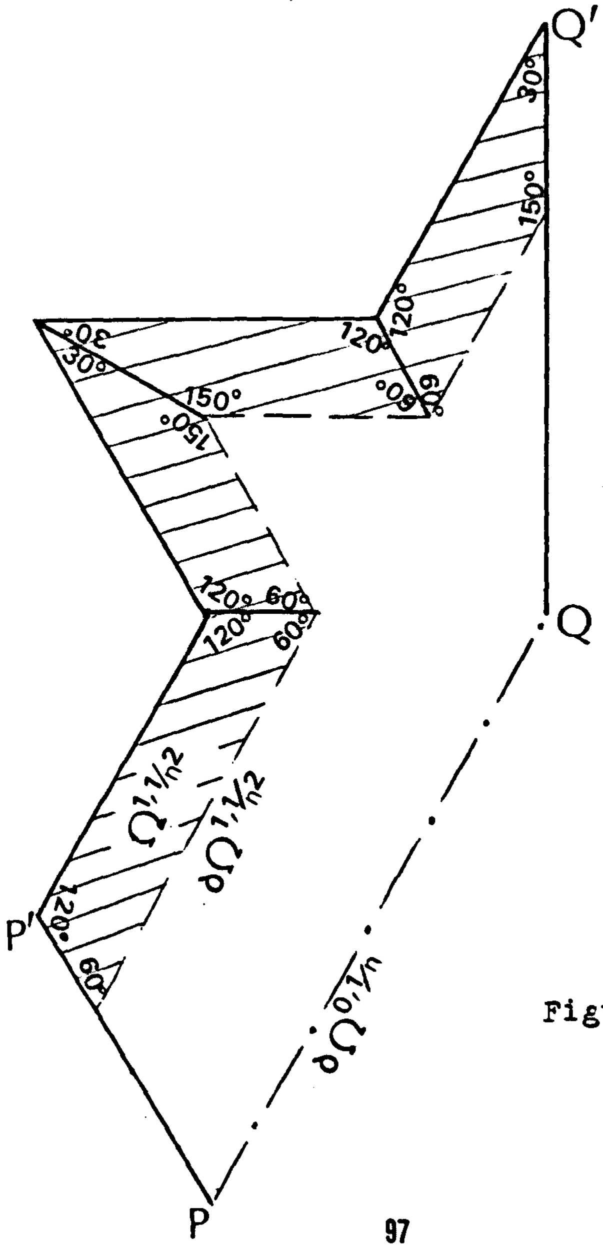


Figure 4

equal parts

a_1, a_2, a_3, a_4 (see figure 3) to be mapped into four equal parts A_1, A_2, A_3, A_4 of $\partial\Omega$.

In taking into account, for the first time, Ω^1 , we also take into account

$$(\Omega^1)^{1/n^2} = \Omega^{1,1/n^2} = \Omega^{1,1/3^2} ,$$

as shown in figure (4), just as we took into account

$$\Omega^o \text{ and } \Omega^{o,1/n}$$

before.

We will extend now the paths from $\partial\Omega^{o,1/n}$ to $\partial\Omega^{1,1/n^2}$, (see figure (4)).

First , we denote with A_1', A_2', A_3', A_4' , the four segments in the section of the boundary $\partial\Omega^{1,1/n^2}$ that we are considering (see figure (5)); we will map a_1 onto A_1' , a_2 onto A_2' , ... etc. Next, we consider an intermediate step between $\partial\Omega^{o,1/n}$ and $\partial\Omega^{1,1/n^2}$, i.e. a segment

$$P''Q'' \subset \partial\Omega^{o,2/n^2} \text{ (see figure (5)),}$$

parallel to the segment PQ (and to P'Q'), and in it, four "intermediate" segments $A_1'', A_2'', A_3'', A_4''$, as seen in figure (5), to be associated with

$$A_1', A_2', A_3' \text{ and } A_4' .$$

And now, we extend the paths, piecewiselinearly, as segments joining points in segments a_i to points in segments A_i'' , for $i \in \{1,2,3,4\}$, as shown in figure (6)(a), and then, in the same way, we extend the paths piecewiselinearly from points in the segments A_i'' to points in segments A_i' , $i \in \{1,2,3,4\}$, as

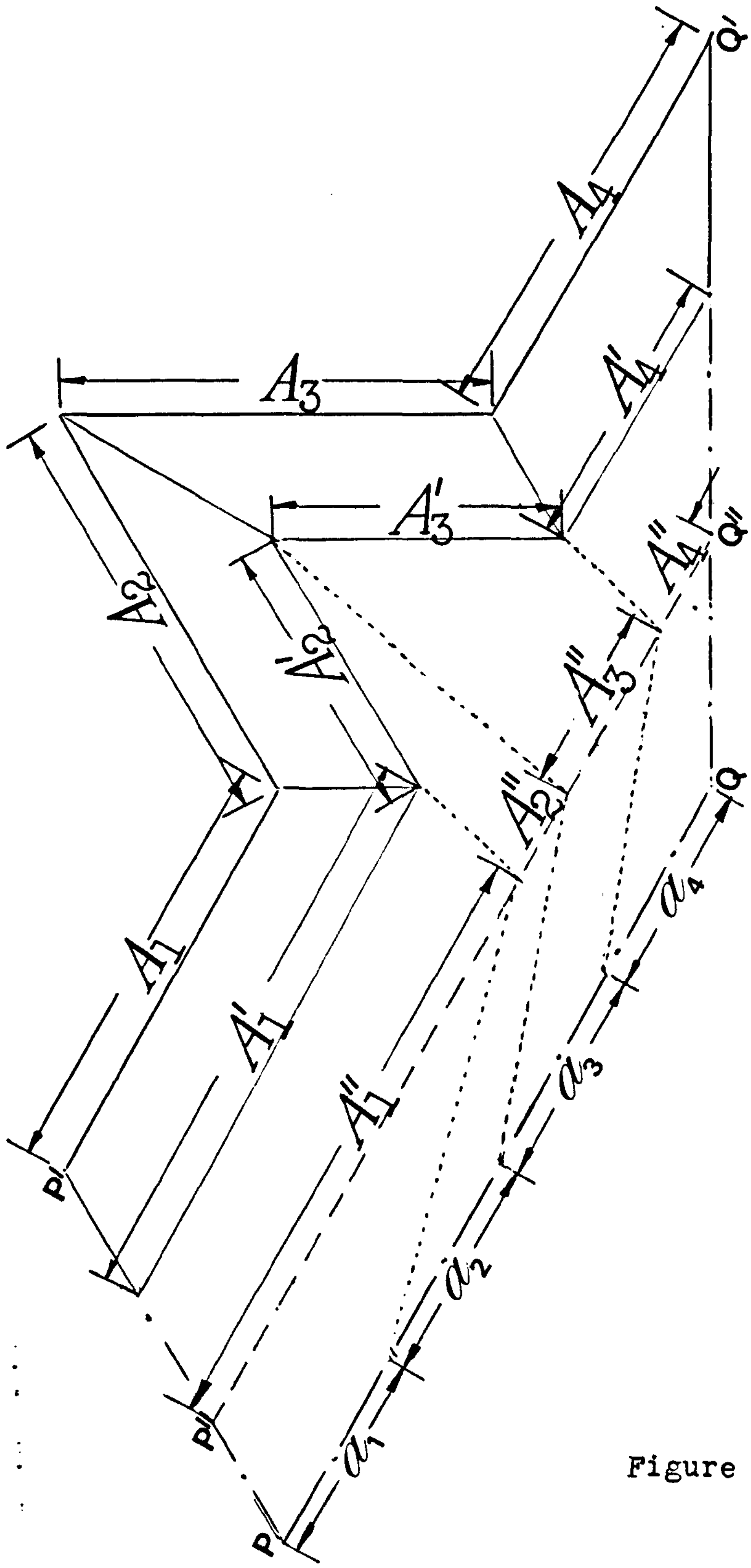
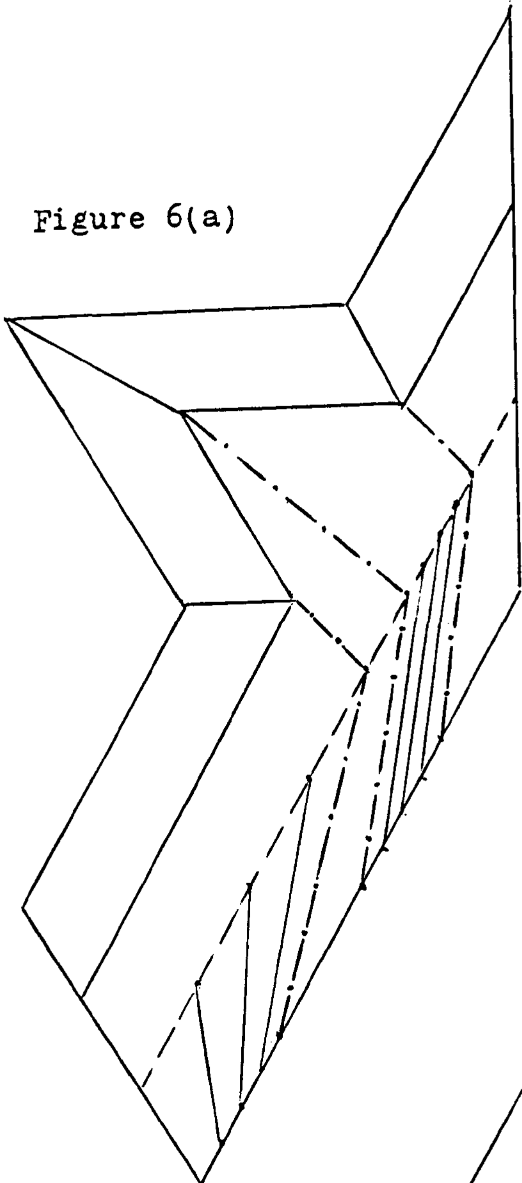


Figure 5

Figure 6(a)



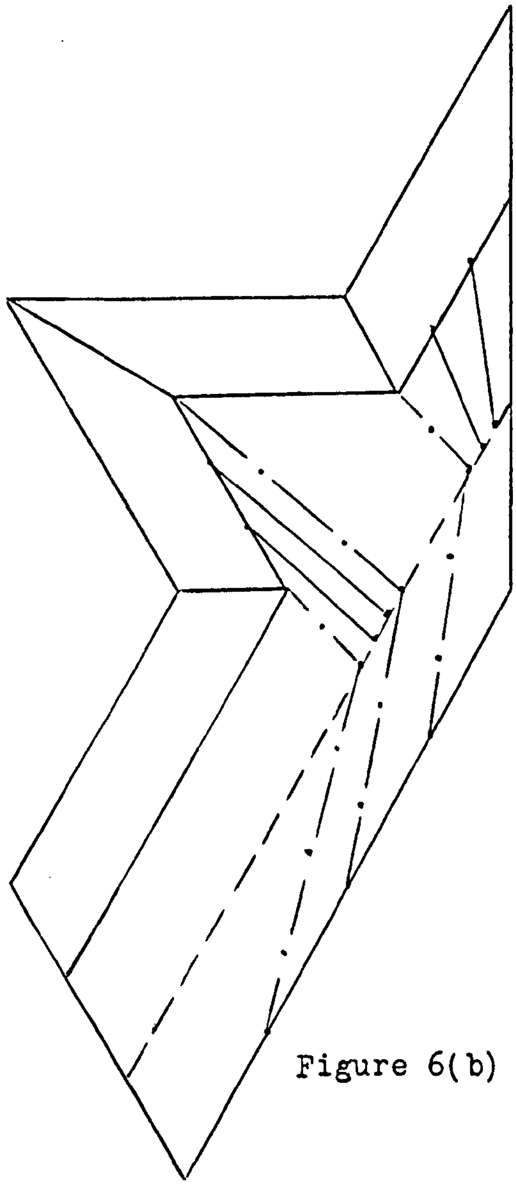


Figure 6(b)

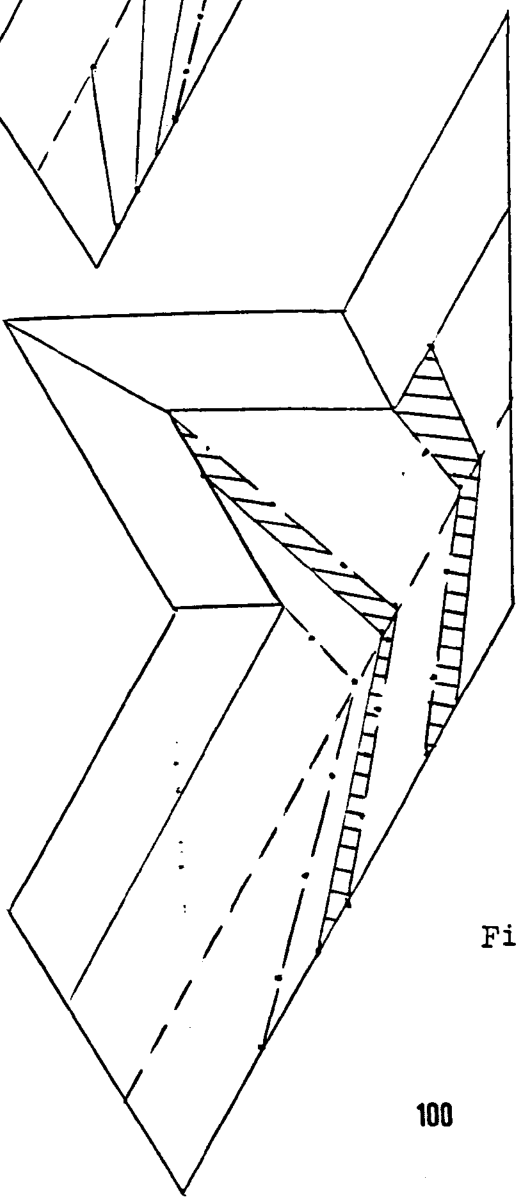


Figure 6(c)

shown in figure (6)(b).

Notice that we have extended our paths , maintaining property **J** (for an example see figure (6)(c)).

The end points of our paths are now in $\partial\Omega^{1,1/n^2}$. In order to extend the paths from $\partial\Omega^{1,1/n^2}$ to $\partial\Omega^{2,1/n^3}$, we first go back to figure (4), and observe that we have the region $\Omega^{1,1/n^2}$ made up by two types of trapezoids : one which is similar to the original one PQQ'P' in $\Omega^{0,1/n}$ (the angles concerned are 30° , 150° , 60° and 120°) which is shown in figure (3) ; and also another one, whose angles are 60° , 60° , 120° and 120° . Let us denote these two trapezoidal shapes by A and B .

Notice that, due to the regularity of the fractal, any $\Omega^{k,1/n^{k+1}}$ will be made up by shapes no other than A and B , reduced in a factor $1/n^k$ (when comparing with the ones in $\Omega^{0,1/n}$), in fact, the case $\Omega^{0,1/n}$ being exceptional in having only the shape A (see figure (7)).

We recall that we had extended our paths from $\partial\Omega^{0,1/n}$ to $\partial\Omega^{1,1/n^2}$ inside a shape A . But to have done it inside a shape B would have entailed no difference of procedure.

We have still the end points of our paths in $\partial\Omega^{1,1/n^2}$, and we next iterate what has been done : we go from $\partial\Omega^{1,1/n^2}$ to $\partial\Omega^{2,1/n^3}$ (and later from $\partial\Omega^{i-1,1/n^i}$ to $\partial\Omega^{i,1/n^{i+1}}$) inside shapes A and B only, by repeating in each shape exactly the procedure described above; thereby mantaining throughout the process the property **J**.

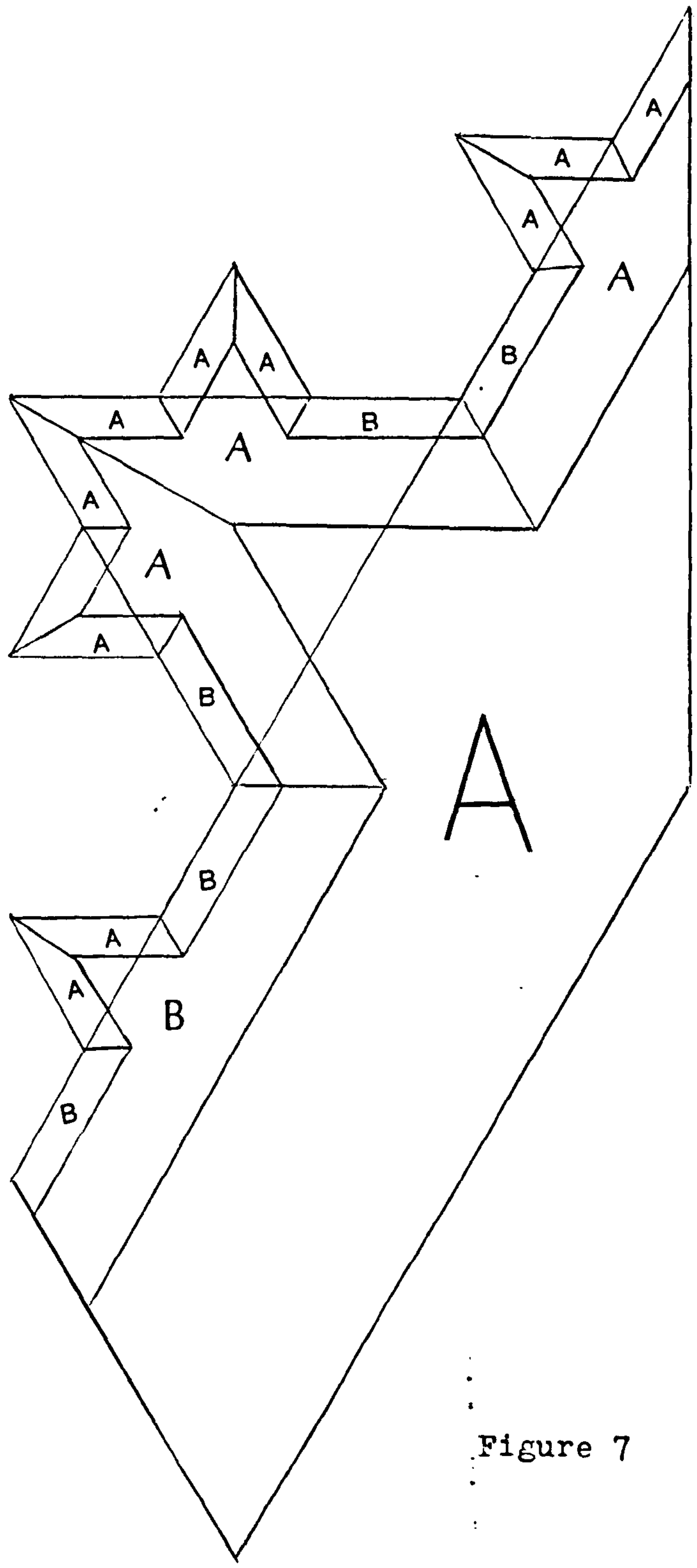


Figure 7

The general case is similar: we extend the paths from $\partial\Omega^{i-1,1/n^i}$ to $\partial\Omega^{i,1/n^{i+1}}$ inside a finite number (this finite number depends on the geometry of the regular fractal) of trapezoidal shapes (instead of only A and B), piecewise linearly, by iterating an initial process that ensures the property $\mathbf{J}$.

By an analogous iterative process of extending the paths from $\partial\Omega^{i-1,1/n^{i+1}}$ to $\partial\Omega^{i,1/n^{i+2}}$ (or from $\partial\Omega^{i-1,1/n^{i+h-1}}$ to $\partial\Omega^{i,1/n^{i+h}}$), again inside a finite number of trapezoidal shapes, we obtain foliations similar to the one just constructed, endowed with basically the same properties.

Notice that, if $A \subset \partial T^1 \subset \partial\Omega$, A a connected set, then we have

$$\mu^1 [\mathbf{P}_{S_0} (A)] = \mu^d (A) \sim [\operatorname{diam}(A)]^d \ .$$

Appendix to Chapter 5

We will construct yet another piecewise linear foliation of the tail T, with similar (but somewhat more interesting) properties (to be pointed out later) to the foliation just constructed. The paths, which will be the leaves of this foliation, will have again initial points on a segment $s_0 \subset \Omega$, and end points on $\partial\Omega$; with the property that segments of equal length on s_0 are mapped, by pushing along the leaves, into sets of equal d-Hausdorff measure in $\partial\Omega$.

Again, we construct these paths in stages,and at each stage, the corresponding portion of the paths will be: line segments joining points of one segment with points of another, in a piecewise linear way.

The starting points of the paths will belong to the segment s_0 already described,the paths will foliate the connected component of $\Omega-\Omega^{\circ}$ to whose boundary s_0 belongs.

As before, we exemplify with the Koch snowflake.

We begin by foliating first the triangle A_1, as indicated in figure (1).

We consider in such region its barycentre β (see figure (2)), and, with β as origin, the region

$$A_1' = \lambda A_1 \;\; ; \;\; \lambda < 1 \quad , \text{(see figure (3))}.$$

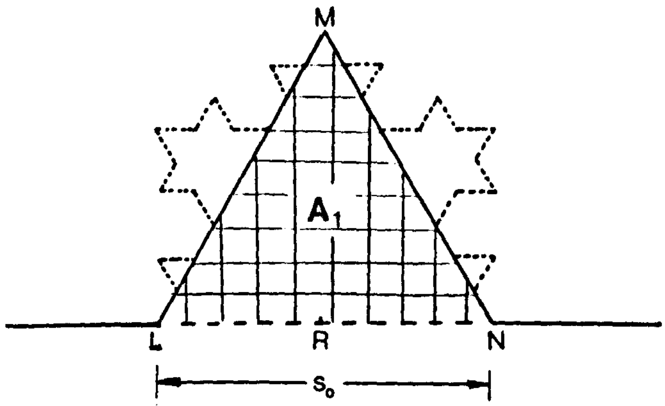


Figure 1

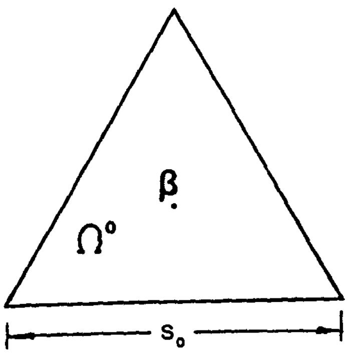


Figure 2

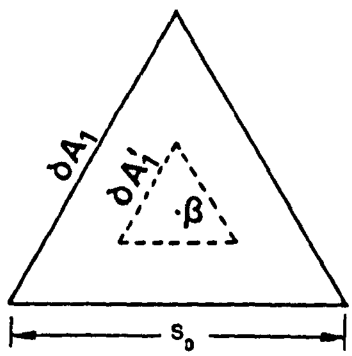


Figure 3

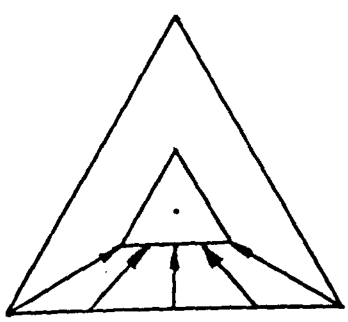


Figure 4

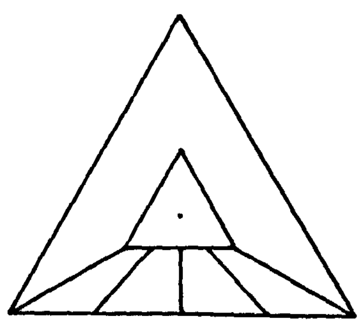


Figure 5

The first stage in the construction of the paths is as follows:

We call s_0' the segment base of the smaller triangle A_1', and we map s_0 into s_0' in a linear way, as shown in figure (4).

We now join poits of s_0 with their images in s_0', by line segments.

These segments are the first stage in the construction of the paths (see figure (5)).

The second stage of the construction of the piecewise linear paths, is as follows: we map the base s_0' of the small triangle A_1', into the other two sides of this triangle, linearly, taking care that equal segments in s_0' are mapped into equal parts of the other two sides, as shown in figure (6).

Again we join, linearly, points of s_0' to their images, and these new segments are the next stage in the construction of the paths.

Notice that, until now, property $\mathbf{J}$ is mantained (see figure (7)); as we extend the paths, we want to mantain that comparability.

We associate segments a and b in $\partial A_1'$, with segments A and B in ∂A_1, as shown in figure (7); and in doing so, we take into consideration, for the first time, Ω^1, as seen in figure (8).

We now divide segment a (and b) into 4 equal segments (a_1, a_2, a_3, a_4 in figures (8) and (9)) to be mapped onto 4 equal parts A_1, A_2, A_3, A_4 (as seen in figure (8)), in order to ensure that "equal parts go to equal parts", and so mantaining throughout property $\mathbf{J}$.

Next, we further distinguish the triangular regions

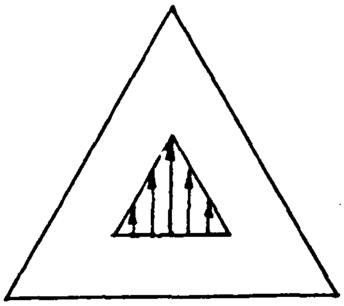


Figure 6

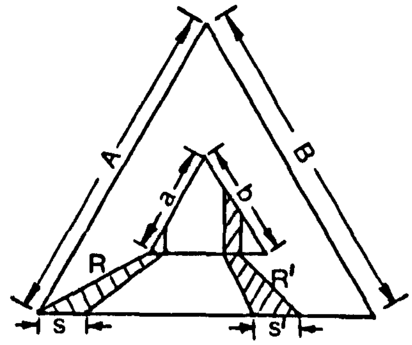


Figure 7

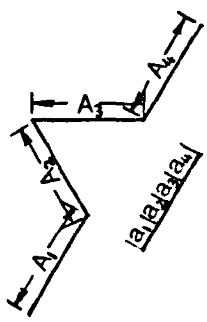


Figure 8

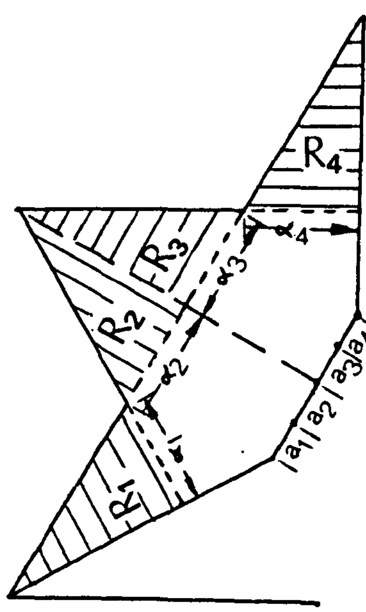


Figure 9

$$R_i \; ; \; R_i \subset \Omega^1 \; ; \; i \in \{1,2,3,4\}$$

as shown in figure (9).

Let α_i be the base of the triangle R_i (see figure (9)); we extend now the paths, piecewise linearly, as segments joining points in segments a_i to points in segments α_i (see figure (10)).

Notice that we mantained the jacobian comparability stated in property $\mathbf{J}$ (see figure (11)).

From now on we iterate what has been done. First we notice that: By comparing figures (1) and (9), we can deduce that, geometrically, subsegments LR and RN (in figure (1)) of s_0, are related to subtriangles LRM and RMN (in the same figure) of A_1, in exactly the same way in which segments α_i relate to triangles R_i, $i \in \{1,2,3,4\}$ (in figure (9)).

Thereby, we will prolong the paths from the segments α_i into the interior of the triangles R_i by repeating in each R_i what has already been done until now in A_1 (see figure (12)). Hence,property $\mathbf{J}$ is mantained.

We iterate this process *ad infinitum*.

The foliation depicted in chapter 5 was based upon considering, inside $\Omega^{\varepsilon} = \Omega^{1/n}$, thinner and thinner $\Omega^{k,1/n^{k+1}}$. This new foliation, instead, is based on the subdivision of the whole tail T into self similar sets B_i^p of decreasing size. We can write

$$T = \sum_{p,i} B_i^p$$

These sets B_i^p are not the self similar triangles A_i^p.

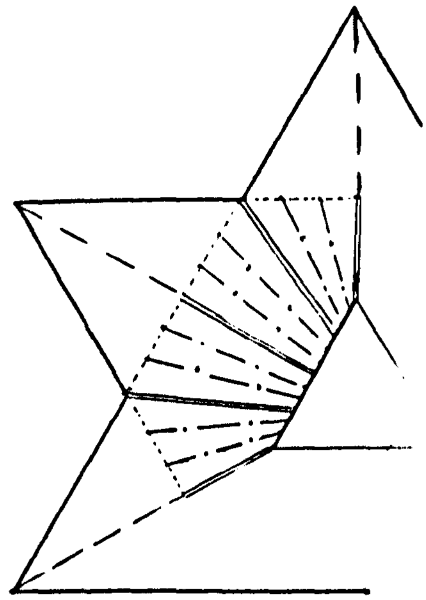


Figure 10

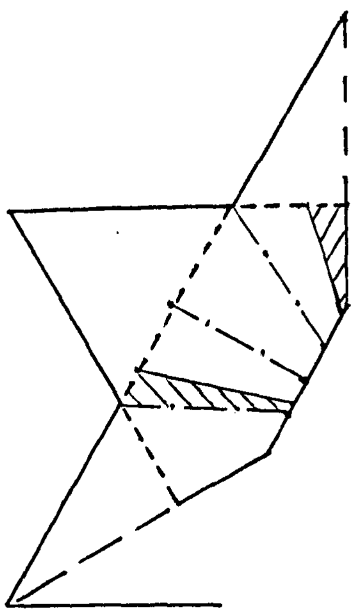


Figure 11

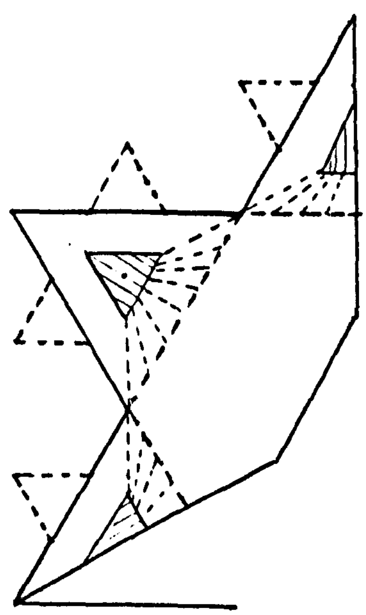


Figure 12

$B_1 = B_1^{\,1}$, for instance, is the triangle LRM (in figure (1)) <u>minus</u> the triangles R_1 and R_4.

<u>All</u> the other B_i^p are similar to B_1.

In each such set B_i^p, $p \in \mathbb{N}$, the leaves of the foliation have the same shape as in B_1.

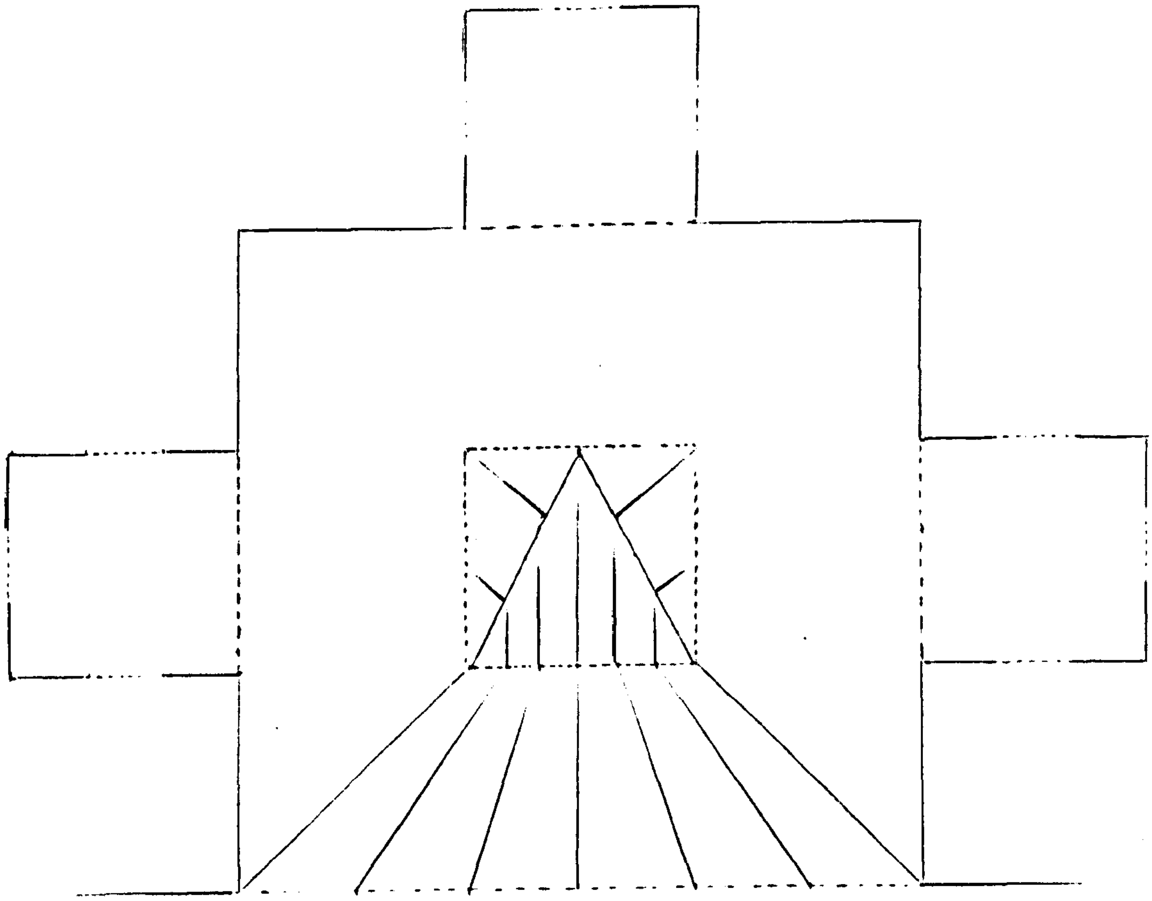
Notice that, the index p in B_i^p stands <u>both</u> for the number of iterations of the foliating process--when constructing the leaves going from s_0 in B_1 to B_i^p -- and for the size of the set B_i^p.

Therefore, for $u \in L^1(T)$, we can write

$$\iint_T u = \sum_{p,i} \iint_{B_i^p} u = \sum_{p,i} \iint_{B_i^p} u_i^p = \sum_{p \in \mathbb{N}} \left(\frac{1}{n^p}\right)^2 \sum_i \iint_{B_1} \tilde{u}_i^p$$

where u_i^p is the restriction of u to B_i^p, and $\tilde{u}_i^p$ is its transformed function corresponding to the 1:1 mapping of B_i^p onto B_1.

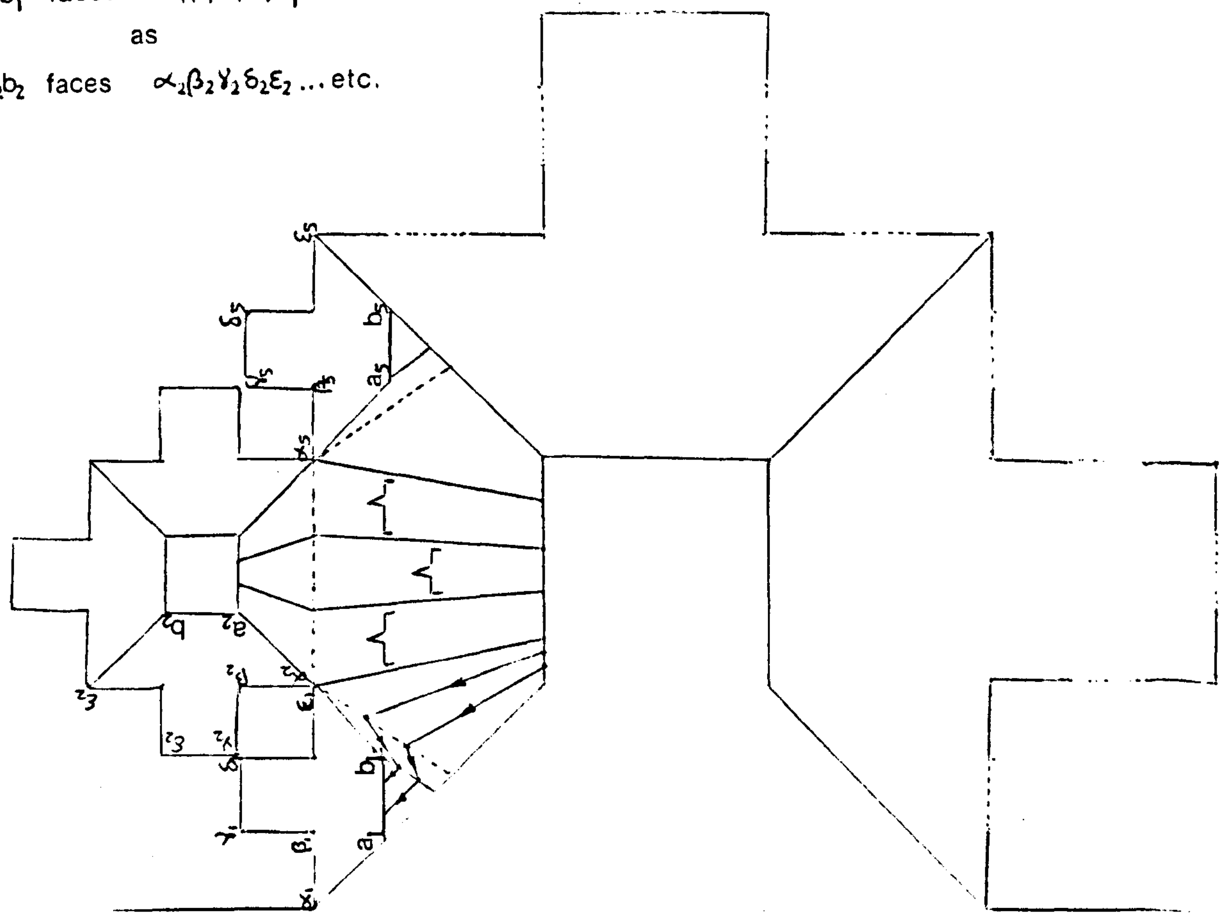
The figures in the next pages give several ideas on how to extend this method of foliation to cases other than the Koch snowflake.

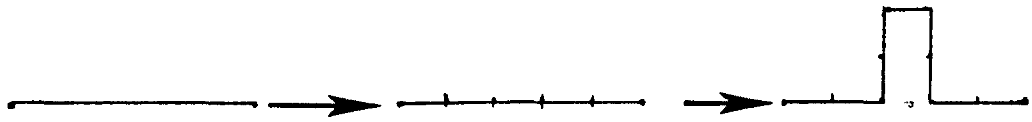


$\overline{a_1 b_1}$ faces $\overline{\alpha_1 \beta_1 \gamma_1 \delta_1 \varepsilon_1}$

as

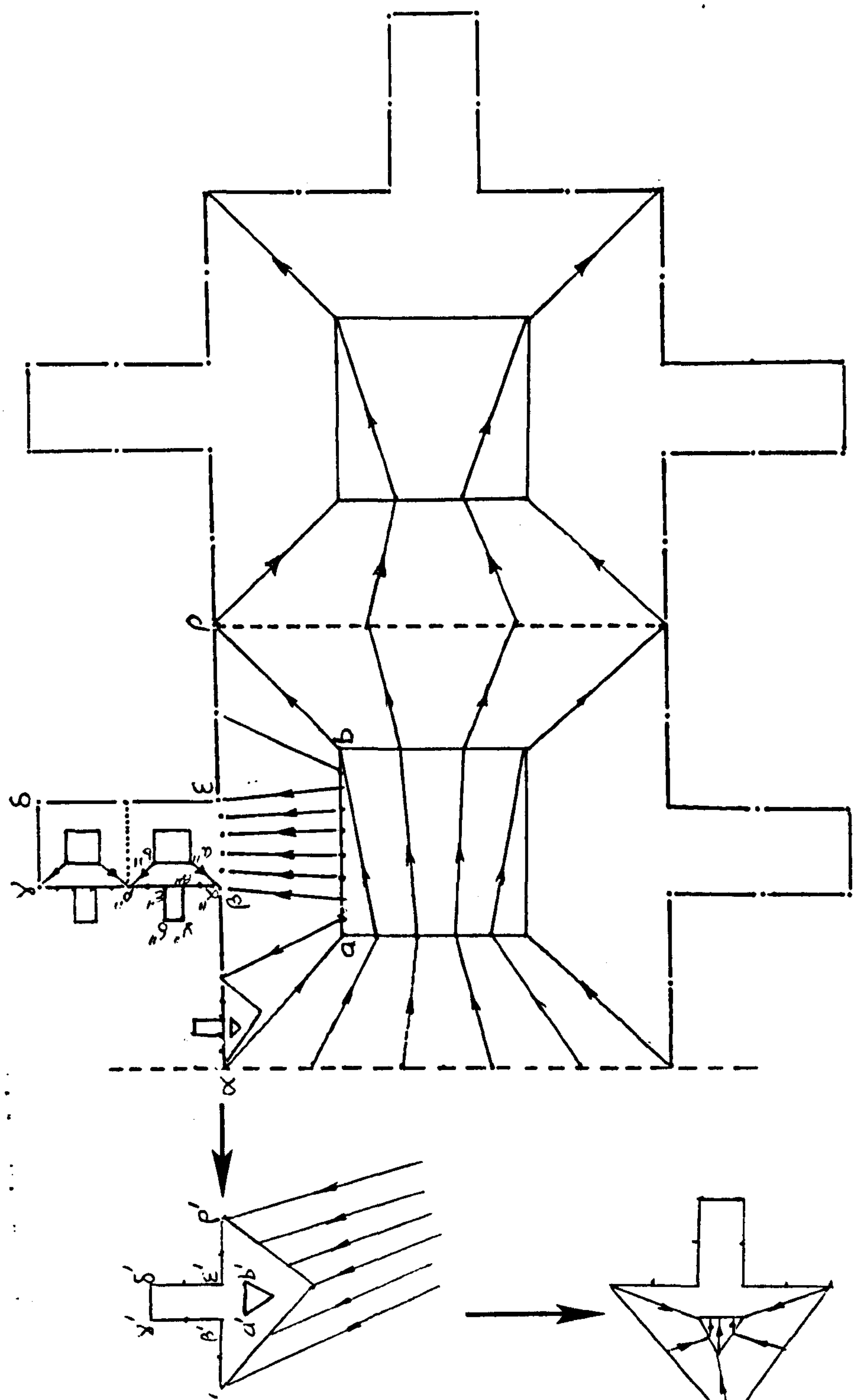
$a_2 b_2$ faces $\alpha_2 \beta_2 \gamma_2 \delta_2 \varepsilon_2$ ... etc.





$\overline{ab}$ faces $\overline{\alpha\beta\gamma\delta\varepsilon\rho}$

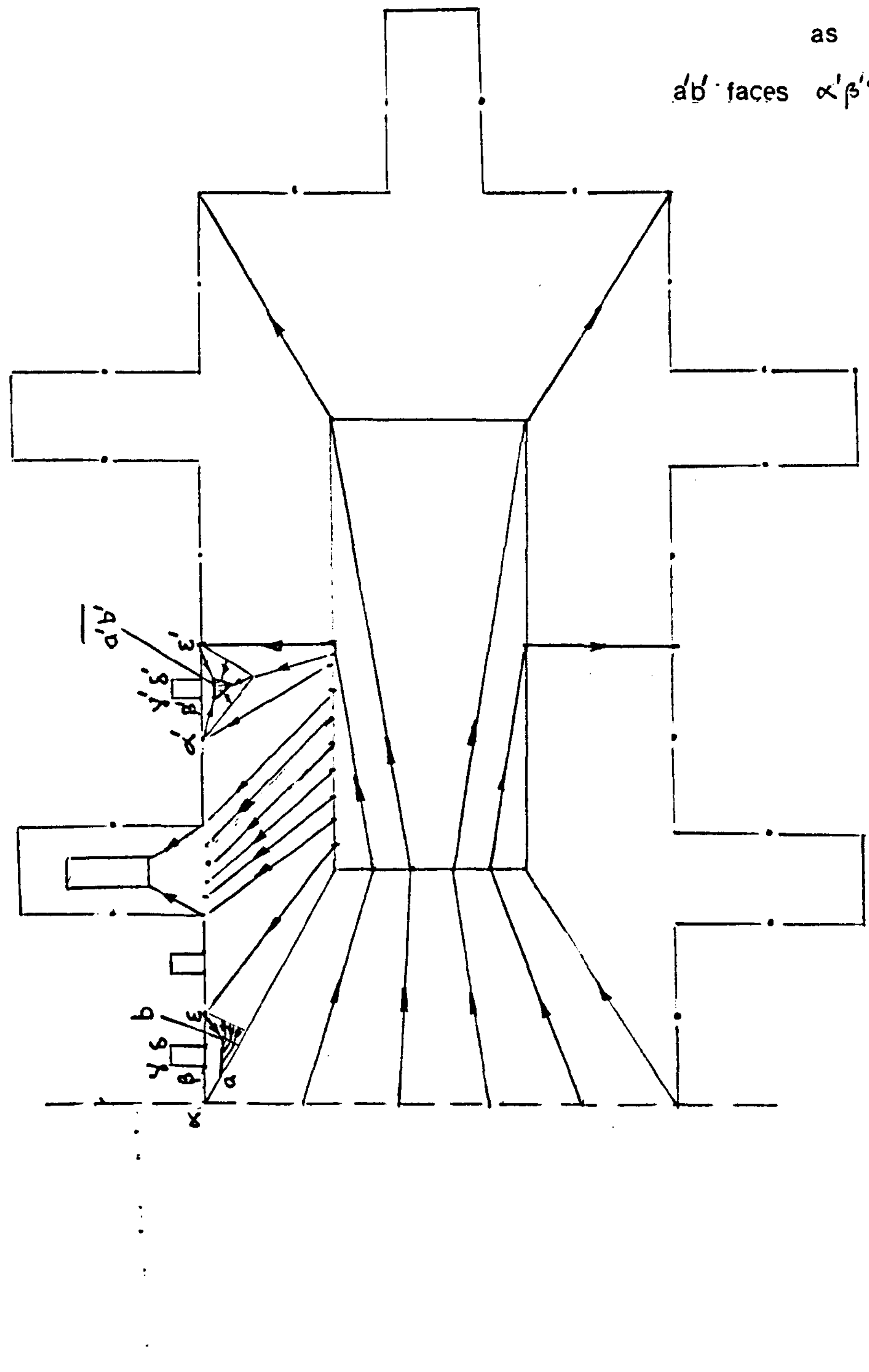
as $a'b'$ faces $\alpha'\beta'\gamma'\delta'\varepsilon'\rho'$ ... etc.



$\overline{ab}$ faces $\overline{\alpha\beta\gamma\delta\varepsilon}$

as

$a'b'$ faces $\alpha'\beta'\gamma'\delta'\varepsilon'$ ...etc.



Chapter 6

In this section we will establish a relationship between the size of a level set $\{ (x,y) \mid u(x,y) \geq h \}$ --in a sense specified below-- and the value of $\mathrm{grad}^2 u$ on the set $u \leq h$.

As before, we will consider u in $L_1^2(\Omega)$, continuous, and supported on a tail T .

We denote: $V^{h'}(u) = \{ (x,y) \mid u(x,y) \geq h' \}$; and $\{ V_\alpha^{h'}(u) \}_{\alpha \in A}$ will denote the set of connected components of $V^{h'}(u)$.

Let $u \geq 0$ be supported on a tail $T^p \subset T$ of size p .

Let $h \in \mathbb{R}^+$; $h = \inf \{ h' \mid$ the sets $V_\alpha^{h'}(u)$, $\alpha \in A$, can be displaced by Euclidean motions and relocated, non rampantly, in a tail T^{p+1} of size $p+1$ in such a way that each $\mathrm{cl}.V_\alpha^{h'}(u) \cap \partial\Omega$, $\alpha \in A$, is relocated in $\partial\Omega \}$.

We will give a name to this particular value of h : the $p \to (p+1)$-relocation value for the function u . When there is no possibility of confusion we will call it the relocation value , and we will say that the connected components $V_\alpha^h(u)$, $\alpha \in A$, "fill up" T^{p+1} .

We will prove that, in this case:

$$\iint_{CV^h(u)} \mathrm{grad}^2 u \geq \alpha(\Omega) h^2 \qquad (1)$$

where $CV^h(u)$ is the set where $u<h$.

Notice that the $V_\alpha^h(u)$ --h the relocation value--can "spill overboard" T^{p+1}, as seen in this example:

supp $u = T^p$; $u=h$ in two tails T^{p+1} contained in T^p; and $u<h$ in the rest of T^p.

But, in the reasonings that will follow, it will make absolutely no difference whether the $V_\alpha^h(u)$ fill up T^{p+1} exactly or spill overboard.

Notice too that the $V_\alpha^h(u)$ can be said to be entirely in $\partial\Omega$, in the sense illustrated by this example:

supp $u = T^p$; $u<h$ in T^p; $u(x,y)\to h$ as (x,y) approaches ∂T^{p+1}; $T^{p+1}\subset T^p$.

We want to avoid this situation, as we prefer to work with sets $V_\alpha^h(u)$ entirely contained in Ω (an open set).

Notice that by working, if necessary, with an arbitrarily small perturbation of h, say, with $h' = h-h/10^M$ -- M arbitrary-- we avoid this situation: all sets $V_\alpha^{h'}(u)$ contained in Ω will "fill up" or "spill overboard" T^{p+1}.

We can work with h' so close to h that we perturb both sides of (1) as little as we want.

It will therefore be enough, in the remainder of the chapter, to work with sets $V_\alpha^h(u)$ contained in Ω.

The following proposition is the equivalent, in one dimension of the relationship (1).

Proposition

Let $f(x) \in L_1^2[0,\Delta]$; $f(0)=a$; $f(\Delta)=a+h$.

Then

$$\int_0^\Delta (f'(x))^2 dx \geq \frac{h^2}{\Delta} \; ,$$

that is, the minimum of the norm of the derivative of all functions with the same endpoints at 0 and Δ is the function joining linearly those endpoints.

Proof : $f(x)$ can be approximated arbitrarily closely in L_1^2 norm by a continuous piecewise linear function (see figure (1)).

For one such a piecewise linear function $P(x)$, let $\{ P_i \}_{i \in I}$ be the induced partition of $[0,\Delta]$, let Δ_i be the length of the interval P_i (see figure (1)(b)), and h_i the increment of the function (and the polygonal) in the same interval, so

$$\sum_{i \in I} h_i = h \;\; , \text{and} \;\; \sum_{i \in I} \Delta_i = \Delta \text{ (see figure (1)(b))}$$

Then,

$$\int (P'(x))^2 \, dx = \sum_{i \in I} \left(\frac{h_i}{\Delta_i}\right)^2 \Delta_i = \sum_{i \in I} \frac{h_i^2}{\Delta_i} = \sum_{i \in I} \left(\frac{h_i}{\Delta_i^{1/2}}\right)^2 \; .$$

But, by Schwartz, we have

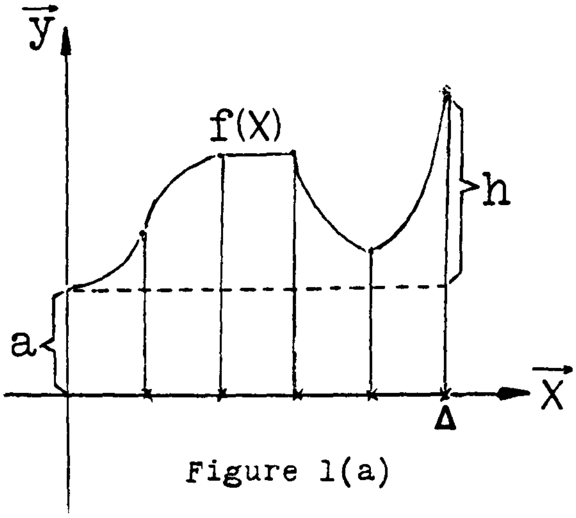


Figure 1(a)

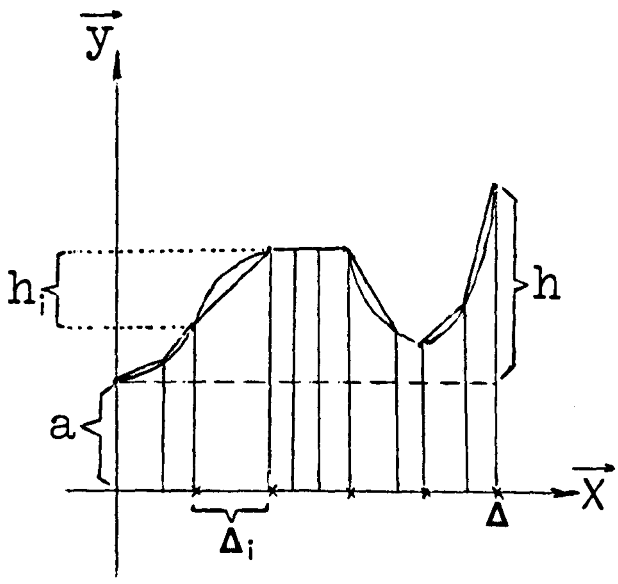


Figure 1(b)

$$h = \sum_{i \in I} h_i = \sum_{i \in I} \frac{h_i}{\Delta_i^{\frac{1}{2}}} \Delta_i^{\frac{1}{2}} \leq \sqrt{\sum_{i \in I} \left(\frac{h_i}{\Delta_i^{\frac{1}{2}}}\right)^2} \sqrt{\sum_{i \in I} (\Delta_i^{\frac{1}{2}})^2} ,$$

so then

$$\sum_{i \in I} \left(\frac{h_i}{\Delta_i^{\frac{1}{2}}}\right)^2 \geq \frac{h^2}{\sqrt{\sum_{i \in I} (\Delta_i^{\frac{1}{2}})^2}^{\,2}} = \frac{h^2}{\Delta} .$$

Next, we will prove a simpler version of the result of this chapter, that we will need in a later section.

Lemma :

Let u be continuous, $u \in L_1^{\,2}(R)$, with ∂R smooth. Let $z_1 < z_2$. Let $V_\alpha^{\,z_1}(u)$ and $V_\beta^{\,z_2}(u)$ belong to $V^{z_1}(u)$ and $V^{z_2}(u)$ respectively; $V_\beta^{\,z_2}(u) \subset V_\alpha^{\,z_1}(u)$.
We will assume $\operatorname{diam}(\partial V_\alpha^{\,z_1}(u) \cap \operatorname{Int} R) \sim \operatorname{diam}(V_\alpha^{\,z_1}(u))$, where the constants of proportionality are fixed and depend only on R .

Then

$$\iint_{V_\alpha^{z_1} - V_\beta^{z_2}} \operatorname{grad}^2 u \geq (z_2 - z_1)^2 \, \frac{\operatorname{diam}(V_\beta^{z_2}(u))}{\operatorname{diam}(V_\alpha^{z_1}(u))}$$

Proof : For brevity, let us rename: $V_\alpha^{\,z_1}(u) = V_1$; $V_\beta^{\,z_2}(u) = V_2$.

a) Let us assume

$$\overline{V_1} \subset \operatorname{Int} R .$$

Let us choose $a \in \partial V_2$; $b \in \partial V_2$ such that the segment ab fulfils

$$\mu^1(\overline{ab}) = \operatorname{diam}(V_2) \ ;$$

and let us consider a local system of coordinates

$$(\vec{x},\vec{y}) \text{ , such that the } \vec{y} \text{ axis passes through } \overline{ab} \text{ .}$$

For every $\bar{y} \in [a,b]$, we consider

$B_{\bar{y}} = \{ (x,y) \in V_1 - V_2 \mid y=\bar{y} \}$,

and for each such B_y , $y \in [a,b]$,we consider just <u>one</u> of its connected components, for brevity let us call it B_y also.

B_y is now a segment in whose endpoints the function u takes the values z_1 and z_2 respectively. Therefore, using our Proposition, for a.e. $y \in [a,b]$ we have:

$$\int_{x \in B_y} \operatorname{grad}^2 u \geq \int_{x \in B_y} u_x^2 \geq \frac{(z_2 - z_1)^2}{\mu^1(B_y)} \geq \frac{(z_2 - z_1)^2}{\operatorname{diam}(V_1)} \ ;$$

and then

$$\iint_{V_1 - V_2} \operatorname{grad}^2 u \geq \frac{(z_2 - z_1)^2}{\operatorname{diam}(V_1)} \int_a^b dy = (z_2 - z_1)^2 \frac{\operatorname{diam}(V_2)}{\operatorname{diam}(V_1)}$$

b) $\partial V_1 \cap \partial R \neq \emptyset$

Let us reduce this case to case a).

Let us consider first a segment s contained in V_1 , and let us suppose, for a moment, that $s = V_2 = \partial V_2$. In case a) we joined points of ∂V_2 to points in $\partial V_1 \cap \operatorname{Int} R$ by a <u>straight</u> line--or segment, which we called B_y . A glance at figure (2) shows that s may be far removed from $\partial V_1 \cap \operatorname{Int} R$, and that we may not be

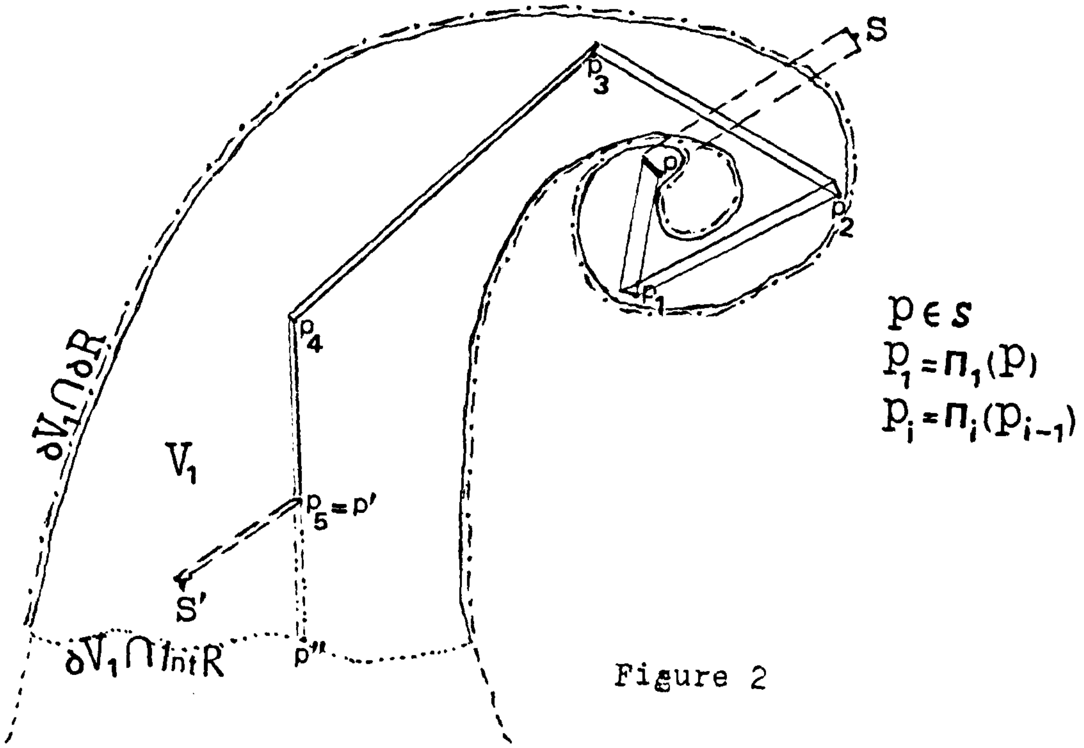


Figure 2

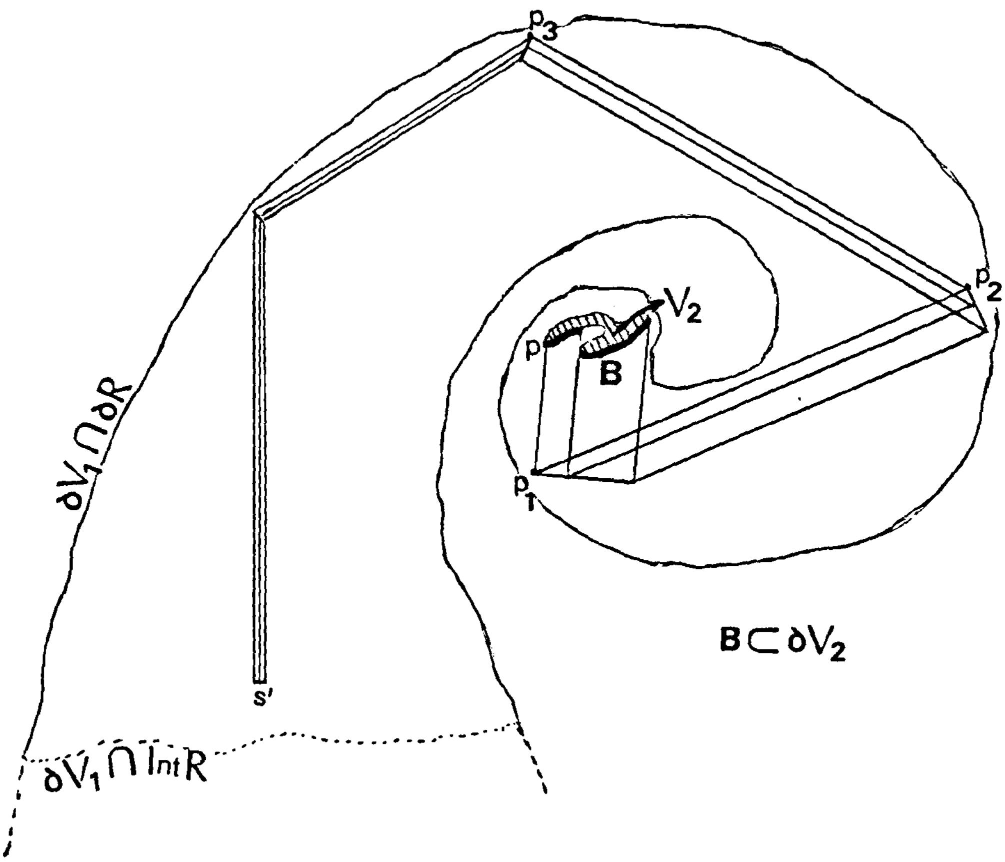


Figure 3

able to join a point $p \in s$ and $\partial V_1 \cap \mathrm{Int}\, R$ with a straight line...but, the smoothness of ∂R implies that we can do it with a broken line of a finite bounded number of segments, as seen in the same figure.

In fact, our aim is to transform s into another segment s' with comparable size, such that s' can be connected to ∂V_1 just as in case a). Our assumption on $\partial V_1 \cap \mathrm{Int}\, R$ will make this possible.

We have, then, a finite number $n = n(R)$ of projections into n directions, $\Pi_1, \Pi_2, \ldots \Pi_n$, such that the composition

$\Pi_n \circ \Pi_{n-1} \circ \ldots \Pi_1(s) =$(definition)

$$\mathop{\circ}_{i=1}^{n} \Pi_i(s) = s'$$

s' a segment, $s' \subset V_1$, and

$$\mu^1(s') = \cos\alpha_n \cos\alpha_{n-1} \ldots \cos\alpha_1 \, \mu^1(s)$$

where α_i is the angle involved in the two consecutive projections Π_{i-1} and Π_i (see figure (2)); $\alpha_i < \pi/2$.

For each point $x \in s$ we have:

Property 1: The segment joining the points

$$\mathop{\circ}_{j=1}^{i} \Pi_j(s)$$

and

$$\mathop{\circ}_{j=1}^{i+1} \Pi_j(s)$$

is entirely contained in V_1;

and the segment s' fulfils

Property 2 : Let $p' \in s'$; $l = l(p')$ a line passing through p' ; $l(p') \perp s'$. Then l contains a segment

$$\overline{p';p''} \text{ , where } p'' \in \partial V_1 \cap \mathrm{Int} R \text{ , and } \overline{p';p''} \subset V_1 \text{ .}$$

see figure (2) for an example.

The very same smoothness of ∂R allows us to extend this result, replacing the segment s by a subset of $\partial V_2 \subset V_1$, namely:

There exists a segment s' with property 2 , such that

$\mu^1(s') \geq \cos\alpha_1 \cos\alpha_2 \dots \cos\alpha_n \operatorname{diam}(V_2) = c(R) \operatorname{diam}(V_2)$ (see figure (3)), and there is a subset $B \subset \partial V_2$ (see the same figure) such that B is mapped onto s' through at most those n projections ($n = n(R)$) $\Pi_1 \dots \Pi_n$, $\Pi_0 = \mathrm{id.}$; $0 \leq \alpha_i < \pi/2$ for every $i \in \{1, \dots, n\}$, and each point $p \in B$ having Property 1 .

Consider, for each $p \in B$, the polygonal arc B_p given by the non rampant sum of segments

$$B_p = \sum_{i \in I(p)} \overline{\mathop{\circ}_{j=1}^{i} \Pi_j(p) \, ; \mathop{\circ}_{j=1}^{i+1} \Pi_j(p)} + \overline{\mathop{\circ}_{i \in I(p)} \Pi_i(p) \, ; p''}$$

where: $\operatorname{card}(I(p)) \leq n = n(R)$,

$$\mathop{\circ}_{i \in I(p)} \Pi_i(p) = p' \in s' \text{ and } p'' \in \partial V_1 \cap \mathrm{Int} R$$

as defined above.

Consider now

$$\sum_{p \in B} B_p = \mathbb{B}$$

and consider a "straightened version" of $\mathbb{B}$, found by mapping each B_p onto a line segment; let us call it $\mathbb{B}'$.

Then there is a mapping $M:\mathbb{B} \to \mathbb{B}'$, the jacobian $\mathbb{J}$ of which fulfils

$$\alpha(R) = \cos\alpha_1 \cos\alpha_2 \ldots \cos\alpha_n \leq \mathbb{J} \leq 1 \quad ,$$

and such that the image of each B_p , $p \in B$, is a segment $B'_p \subset \mathbb{B}'$, and this segment has endpoints in which u' (the function corresponding to u in the mapping $M:\mathbb{B} \to \mathbb{B}'$) takes the values z_1 and z_2 .

We are then in case a).

Note : There may be other connected components $V_\gamma^{z_2}$, $V_\delta^{z_2}$, ...etc (<u>besides</u> V_2) contained in V_1. Let us modify the function u <u>inside</u> those extra components; the modified function u' will have the constant value z_2 in all those $V_\gamma^{z_2}$, $V_\delta^{z_2}$, ...etc (see figure (4)(b)); and u' will be continuous and in $L_1^2(R)$.

We have: $\mathrm{grad}^2\, u' = 0$ inside $V_\gamma^{z_2}$, $V_\delta^{z_2}$, ... ;

also: in the set $S = (V_1 - V_2) - V_\gamma^{z_2} - V_\delta^{z_2} - \ldots$ we have:

$\mathrm{grad}^2\, u' = \mathrm{grad}^2\, u$.

The Property proved before the lemma needs <u>only</u>

$$u'\big|_{\partial V_2} \geq z_2 \text{ ,and } u'\big|_{\partial V_1 \cap \mathrm{Int}\, R} = z_1$$

(which is true for u' as much as for u) in order to yield (by repeating the reasoning of the lemma):

$$\iint\limits_{V_1 - V_2} \mathrm{grad}^2\, u' \geq (z_2 - z_1)^2 \frac{\mathrm{diam}\, V_2}{\mathrm{diam}\, V_1}$$

But the last integral is equal to

$$\iint\limits_{S} \mathrm{grad}^2\, u = \iint\limits_{\substack{V_1 - V_2\ ; \\ (z_1 \leq u \leq z_2)}} \mathrm{grad}^2\, u$$

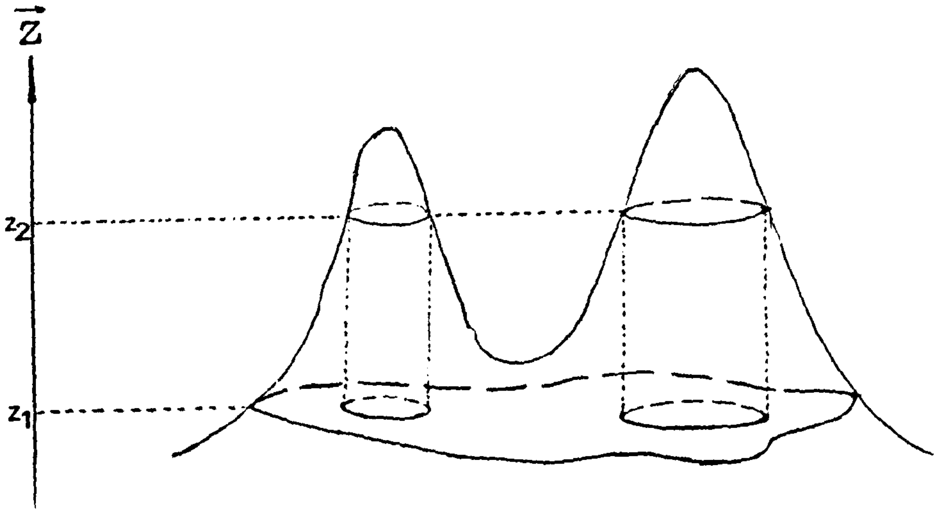


Figure 4(a)

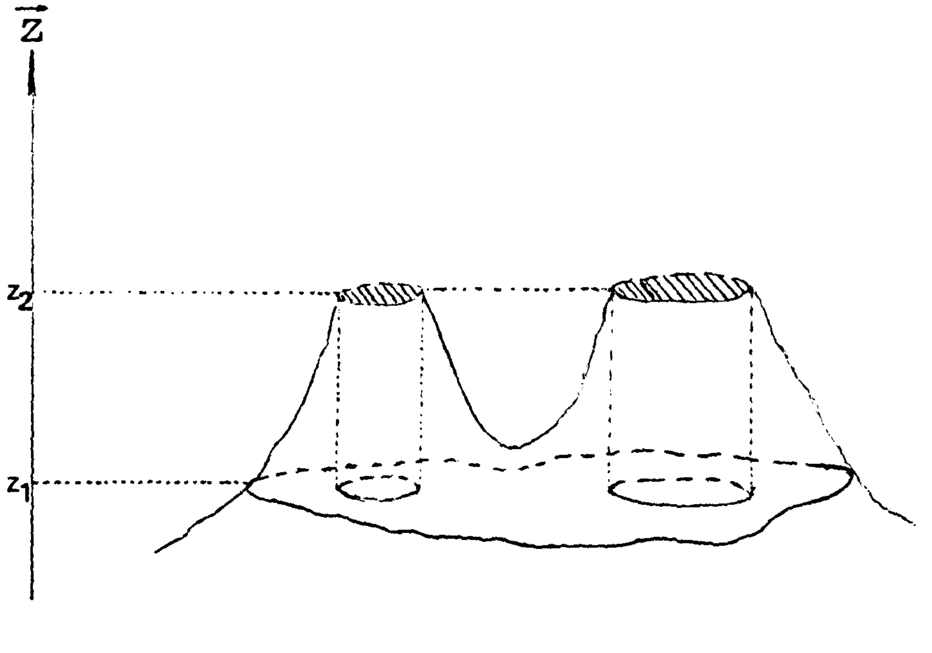


Figure 4(b)

and then we have:

Lemma' : Under the same hypotheses as the previous lemma, we have:

$$\iint_{\{z_1 \leq u \leq z_2\}} \operatorname{grad}^2 u \geq (z_2 - z_1)^2 \frac{\operatorname{diam} V_2}{\operatorname{diam} V_1}$$

Notice that, if ∂R is not smooth, but it has, for instance, a horn of infinite turns, then, instead of a maximum of $n = n(R)$ projections--as indicated in the proof of the lemma--, that number n would depend now on the size of $\operatorname{diam}(V_2)$ or on $\mu^1(s)$, (s is a segment as in the part a) of the proof of the lemma) for, if s is very small, it can be located far enough inside the horn as to need an arbitrarily large number of projections to map it out of the horn and onto s'.

However, by applying the same methods used in the lemma, it is easy to see that

Corollary : If $\partial R = \partial\Omega$ is a regular fractal, if u , V_1 , V_2 , $z_1 < z_2$ are all as defined above, and if

$$\operatorname{diam}(V_2) \geq c(\Omega) \operatorname{diam}(V_1) ,$$

we then have:

$$\iint_{\substack{V_1 - V_2 ;\\ \{z_1 \leq u \leq z_2\}}} \operatorname{grad}^2 u \geq c(\Omega)\,(z_2 - z_1)^2$$

Let us go back now to the problem stated at the begining of the chapter for $u \geq 0$, $\operatorname{supp} u \subset T_p \subset T$; z_1 is 0 , z_2 is h , the relocation value.

The corollary tells us that the problem of proving estimate (1) is solved, if just <u>one</u> of the connected components of the set $V^h(u)$ has size comparable to the size of T_p; the constant of proportionality in (1) will depend on this comparability.

Thus, it remains only to prove the relationship (1) for the case in which <u>all</u> the sets in $V^h(u)$ are of diameter no bigger than $1/n^{p+7}$ --this value is chosen for the case of the snowflake, for reasons which become clear later, the corresponding choice for other cases will be equally clear.

The sets belonging to $V^h(u)$ are of two types: either their clausures have a point in common with $\partial\Omega$, or they do not, accordingly, we will regroup them in sets $\mathbb{A}$ and $\mathbb{B}$ respectively:

$$V^h(u) = \mathbb{A} \cup \mathbb{B} \ ; \ \mathbb{A} \cap \mathbb{B} = \emptyset \ .$$

Essentially, the pieces in $\mathbb{B}$ can be relocated in <u>any place inside</u> T^{p+1}; whereas the pieces in $\mathbb{A}$ can <u>only</u> be relocated touching the boundary of T^{p+1}, specifically, in $\Omega^{1/n^{p+7}} \cap T^{p+1}$.

We need:

Definition : Let $s(x)$ be the leaf--in the foliation of the tail T^p depicted in Chapter 5--with <u>starting point</u> $x \in s_p$. Let S be a set contained in T^p. The projection of S on s_p is, by definition:

$$\mathbb{P}_{s_p}(S) = \{x \in s_p \mid s(x) \cap S \neq \emptyset\}$$

The definition of $V^h(u)$ given above implies that, in a sense, the pieces in $V^h(u)$ "fill up" T^{p+1} --for any other $h'<h$ the sets in $V^{h'}(u)$ "spill overboard" T^{p+1} --. Then, either the set $\mathbb{A}$ or the set $\mathbb{B}$ (or both) is responsible for that "filling up". Hence, in what follows, we can deal separately with the cases $V^h(u)= \mathbb{A}$ and $V^h(u)= \mathbb{B}$.

We will prove:

Claim 1: Let $V^h(u)= \mathbb{B}$. Then, there exists $c=c(\Omega)$ such that

$$\mu^1[\mathbb{P}_{s_p}(\mathbb{B})] \geq \alpha(\Omega)\, \mu^1(s_p)$$

that is, the projection of $\mathbb{B}$ on s_p cannot be arbitrarily small.

Claim 2: Let $V= \mathbb{A}$. Then, there is another u' for which

$$\iint_{u \leq h} \mathrm{grad}^2\, u \sim \iint_{u' \leq h} \mathrm{grad}^2\, u' \ ,$$

and the corresponding set $\mathbb{A}'$ fulfills

$$\mu^1[\mathbb{P}_{s_p}(\mathbb{A}')] \geq \alpha(\Omega)\, \mu^1(s_p)$$

Proof of claim 1 : Let us denote:

$$\{ \Omega^{p+5,\, 1/n^{p+6}} \cup \ldots \cup \Omega^{p+5+i,\, 1/n^{p+5+i+1}} \cup \ldots \} \cap T^p = S \, ,$$

S is so small that all the (even smaller) elements of $V^h(u)$ intersecting S can be easily relocated in a <u>small</u> subset of T^{p+1} (in the snowflake, a simple modification of the argument is needed in other cases); so the pieces in $V^h(u)$ responsible for "filling up" T^{p+1} must be the ones intersecting $T^p - S$, ...and we can restrict ourselves to those.

For the sake of simplification we relocate first the pieces in $V^h(u)$. The small size of the elements in $V^h(u)$, their being away from the boundary, and the properties of our foliation imply that the sets $V_\alpha{}^h(u)$, $\alpha \in A$, can be translated and non rampantly relocated in T^p (see figure (5) for an example) in such a way that

$$\text{a)} \quad \mu^1[\mathbb{P}_{s_p}\{\sum_{\alpha\in A}[V^h_\alpha(u)]'\}] \sim \mu^1[\mathbb{P}_{s_p}\{\sum_{\alpha\in A} V^h_\alpha(u)\}]$$

$$\text{b)} \quad \mathbb{P}_{s_p}\{\sum_{\alpha\in A}[V^h_\alpha(u)]'\} \subset [a,b] \subset s_p \quad ;$$

where $[a, b]$ is an interval such that

$$\mu^1([a,b]) \sim \mu^1[\mathbb{P}_{s_p}\{\sum_{\alpha\in A} V^h_\alpha(u)\}]$$

where $[V_\alpha{}^h(u)]'$ is the relocation in T^p of $V_\alpha{}^h(u)$ (we simply pushed the pieces $V_\alpha{}^h(u)$ together).

We will work with the relocations $[V_\alpha{}^h(u)]'$, which will be denoted simply by $V_\alpha{}^h(u)$ for the sake of brevity.

We can see that, if $[a, b]$ is small enough, then our (small) pieces in $V^h(u)$, contained between two leaves of our foliation (two polygonals of five sides each) as close as we want them, will be relocatable inside T^{p+1} in a very small set ... but then the elements in $V^h(u)$ do not at all fill up T^{p+1}, absurdum.

Proof of claim 2 : In order to simplify the notation we will write $V_\alpha{}^h(u) = V_\alpha$ when there is no danger of confusion.
The pieces in $\mathbb{A}$ can have a very different nature than those in $\mathbb{B}$.

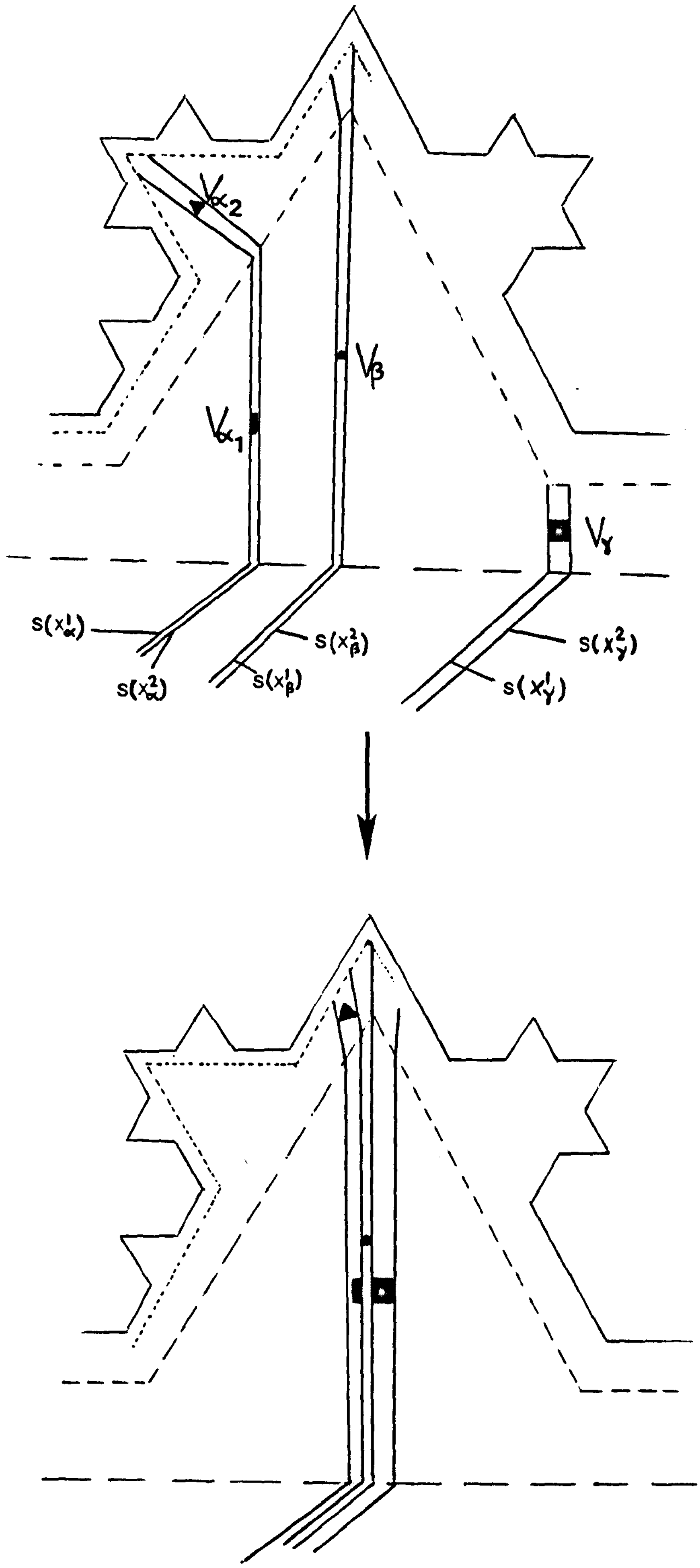

V_{α_2}
V_β
V_{α_1}
V_γ
$s(x^1_\alpha)$
$s(x^2_\alpha)$
$s(x^2_\beta)$
$s(x^1_\beta)$
$s(x^2_\gamma)$
$s(x^1_\gamma)$

Figure 5

Let us focus on the worst possible case: the V_α considered in figure (6) is a subset of a leaf of our foliation with $P \in \partial\Omega$ as the ending point. V_α coils like a horn an infinity of times.

Notice that we could not relocate this particular V_α inside Ω with endpoints P', P'', P''', or P^{iv} (in figure (6)): in fact, V_α may be of such nature that we could not possibly have two of such sets between $\mathbb{P}$ and $\mathbb{Q}$ in figure (6), ... in a way, it is as if V_α "occupied" the whole of ∂T^p between $\mathbb{P}$ and $\mathbb{Q}$.

Consider the smallest value of i for which

$V_\alpha \cap \partial\Omega^{i, 1/n^{i+2}} \neq \emptyset$ (see figure (6)); let us call it $i(V_\alpha)$.

For the sake of brevity the corresponding $\partial\Omega^{i-1, 1/n^{i+1}}$, $i = i(V_\alpha)$, will simply be denoted by $\partial\Omega_\alpha$. In the example in figure (6) we can see that the projection $\mathbb{P}$ of V_α on $\partial\Omega_\alpha$ is just one point. Hence $\mathrm{diam}[\,\mathbb{P}_{\partial\Omega_\alpha}(V_\alpha)\,] = \mu^1[\,\mathbb{P}_{\partial\Omega_\alpha}(V_\alpha)\,] = 0$, and the same happens with $\mu^1[\,\mathbb{P}_{s_p}(V_\alpha)\,]$. This means that the size of such a V_α has no relationship with $\mu^1\{\,\mathbb{P}_{s_p}(V_\alpha)\}$ or with $\mu^1[\,\mathbb{P}_{\partial\Omega_\alpha}(V_\alpha)\,]$. (Notice, nevertheless, that T^{p+1} can be "filled up", in the sense of relocation explained above, with such sets V_α of this same shape and size: if $\mu^1(\,[\mathbb{P}, \mathbb{Q}]\,) = 1/n^{p+h}$, then T^{p+1} has room for just N^{h-1} of such V_α).

We will distinguish, therefore, two cases: either the size of $\mathbb{P}_{\partial\Omega_\alpha}(V_\alpha)$--that is $\mathrm{diam}[\,\mathbb{P}_{\partial\Omega_\alpha}(V_\alpha)\,]$--has a certain relationship with the size of V_α or it has not: in the case of the snowflake, for instance, either $\mathrm{diam}[\,\mathbb{P}_{\partial\Omega_\alpha}(V_\alpha)\,] \geq \mathrm{diam}(V_\alpha)/N^2$ (1)
or it is not. *(The idea is to separate the* V_α "perpendicular" *to the paths (or "parallel" to* $\partial\Omega$*, e.g. as if* $V_\alpha \subset \partial\Omega_\alpha$ *or* $V_\alpha \subset \partial\Omega$*), from those* V_α *(like the one in Fig.6)* "parallel" *to them).*

We will group the corresponding $V\alpha$ into two

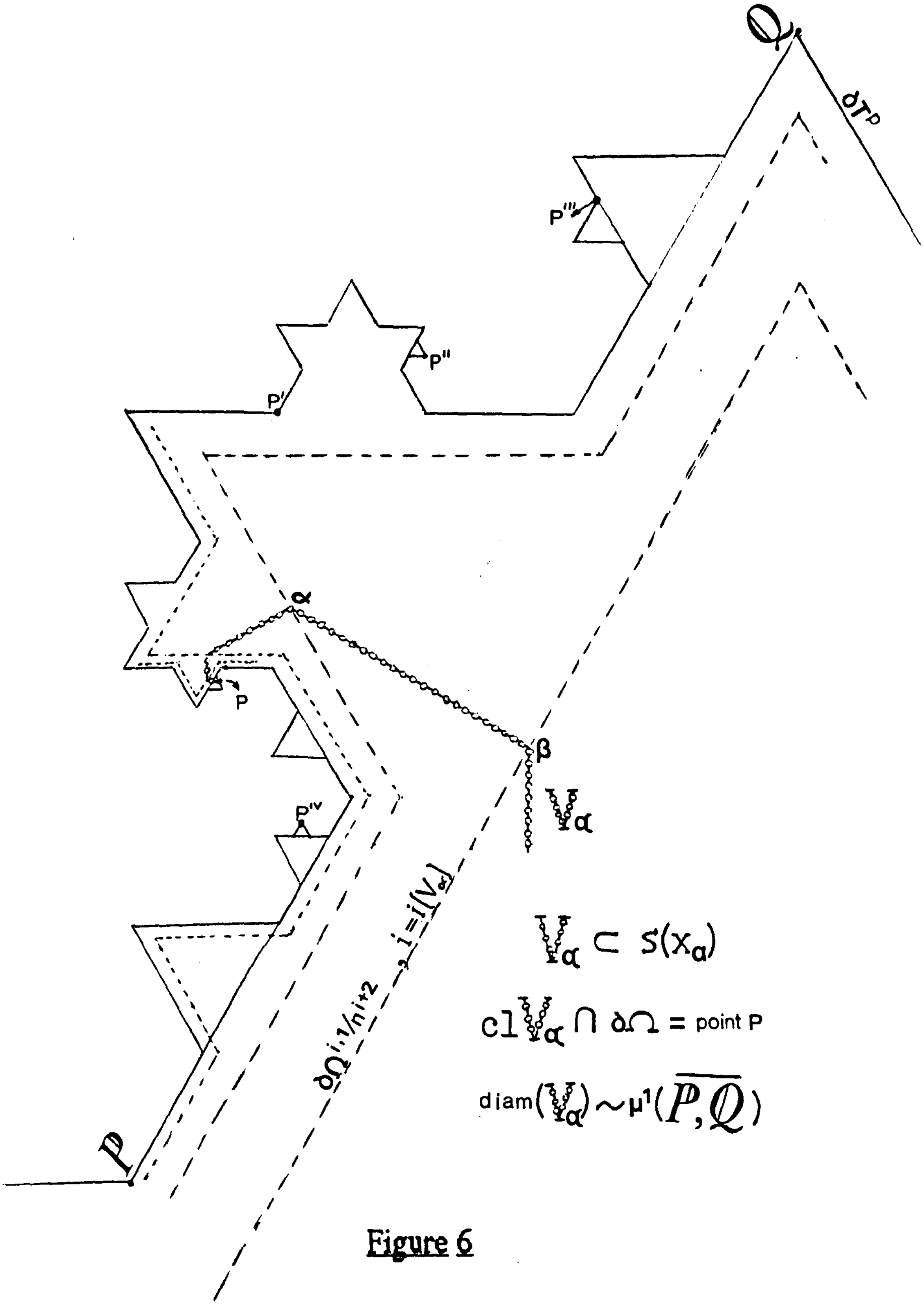


Figure 6

For the sake of clarity, in figures 6, 7, 8, and 9, we draw the paths as being extended from $\partial\Omega^{i-1,1/n^{i+1}}$ to $\partial\Omega^{i,1/n^{i+2}}$, $i \in \mathbb{N}$.

corresponding subsets: $\mathbb{A}_1$ and $\mathbb{A}_2$.

Let us suppose that $\mathbb{A} = \mathbb{A}_1$.

First we observe that, for $V_\alpha \in \mathbb{A}_1$, *the very nature of the construction of the paths ensures that (1) implies*

$\mu^1\{\mathbb{P}_{s_0}(V_\alpha)\} \geq c\,(\mathrm{diam}V_\alpha)^d$, ...exactly as if--except for a multiplicative constant--we had $V_\alpha \subset \partial\Omega$ (1')

Let us denote with V_α^* the relocation in $T^{p+1} \cap \Omega^{1/n^{p+5}}$ of the set V_α . Now, the V_α^* "fill up" T^{p+1} (for $\mathbb{A} = V^h(u)$) ... therefore, from (1') it is easy to see that

$$\mu^1[\mathbb{P}_{s_p}\{\sum_{\alpha\in A} V_\alpha^* \}] \sim \mu^1[\mathbb{P}_{s_p}(T^{p+1})] = 1/N\,\mu^1(s_p)$$

and then to conclude that our claim 2 would be proved by choosing $u' = u$.

Let us suppose now that $\mathbb{A} = \mathbb{A}_2$.

Each V_α is now not very different from the type depicted in figure (6), for its projection on $\partial\Omega^{i\,,\,1/n^{i+2}}$ is, likewise, very small--when compared with the size of V_α .

In order to tackle this problem we introduce the transformation M shown in figure (7). It basically consists of a smooth deformation of *each* $\Omega^{i,\,1/n^{i+2}}$ onto itself, hence, of S--as defined in the statement of claim 2--onto itself.

All leaves $s(x)$ of our foliation are now deformed accordingly in *each* $\Omega^{i,\,1/n^{i+2}}$ (see figure (7)(b)).

M fulfills the hypotheses in the Lemma of chapter 3 .

Let us now focus on figure (7)(b). The magnitude of the deformation--introduced by M--of the paths $s(x)$ in any $\Omega^{i,\,1/n^{i+2}}$ is shown in that figure in the two quantities a and b .

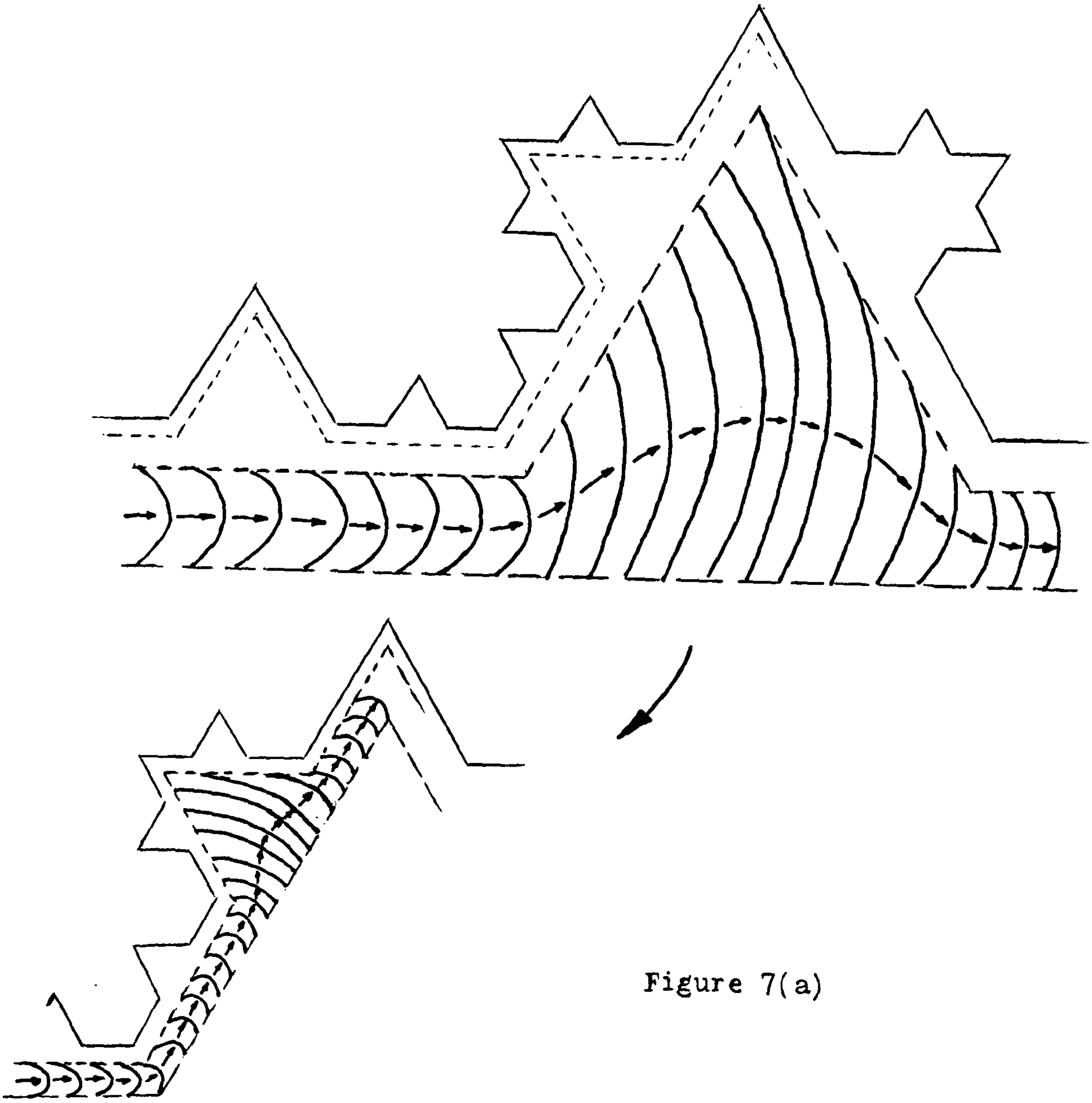


Figure 7(a)

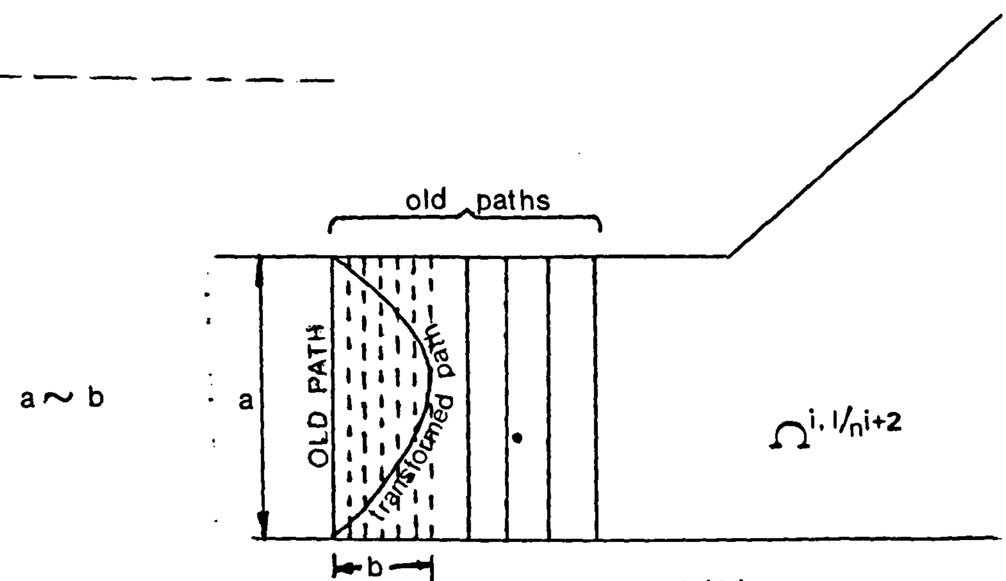


Figure 7(b)

The chosen proportion relating a and b -- $a \sim b$ -- implies: each transformed path intersects so many of the old ones as to ensure:

If $V_\alpha \in \mathbb{A}_2$, and V_α' is the set transformed by M, then,

$\mu^1\{\mathbb{P}_{\partial\Omega_\alpha}(V_\alpha')\} \sim \operatorname{diam} V_\alpha \sim \operatorname{diam} V_\alpha'$, which is the result equivalent to (1) (the constants of proportionality do not depend on the particular set V_α). An example is worked out in figure (8) for the V_α defined above in figure (6).).

Therefore, repeating here the reasoning used in the case $\mathbb{A} = \mathbb{A}_1$, we obtain also for the V_α' the result equivalent to (1'), ...and repeating the rest of the same reasoning we conclude

$$\mu^1[\mathbb{P}_{s_p}\{\sum_{\alpha\in A}[V_\alpha^*]'\}] \sim \mu^1[\mathbb{P}_{s_p}(T^{p+1})] = 1/N\,\mu^1(s_p)$$

...and the proof is finished if we notice that

$$\mu^1[\mathbb{P}_{s_p}\{\sum_{\alpha\in A} V_\alpha'\}] \geq c\,\mu^1[\mathbb{P}_{s_p}\{\sum_{\alpha\in A}(V_\alpha^*)'\}]$$

(see figure (9)), and if we set $u' = u \circ M$.

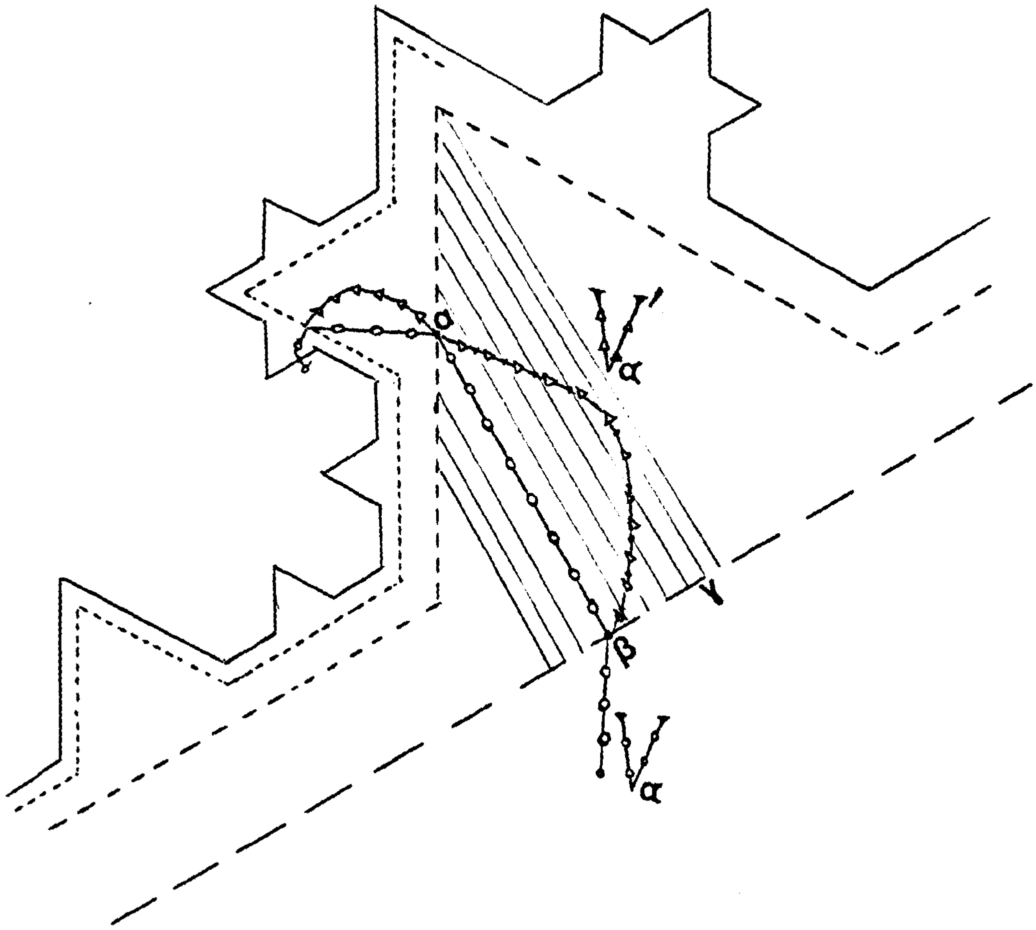
Now, in the general case $\mathbb{A} = \mathbb{A}_1 + \mathbb{A}_2$, either we have

$$\mu^1[\mathbb{P}_{s_p}\{\sum_{V_\alpha\in\mathbb{A}_1} V_\alpha\}] \geq \frac{1}{N^2}\mu^1(s_p)$$

or not.

In the first case our claim is proved by writing $u' = u$ --for then we would have

$$\mu^1[\mathbb{P}_{s_p}\{\sum_{V_\alpha\in\mathbb{A}} V_\alpha(u')\}] =$$



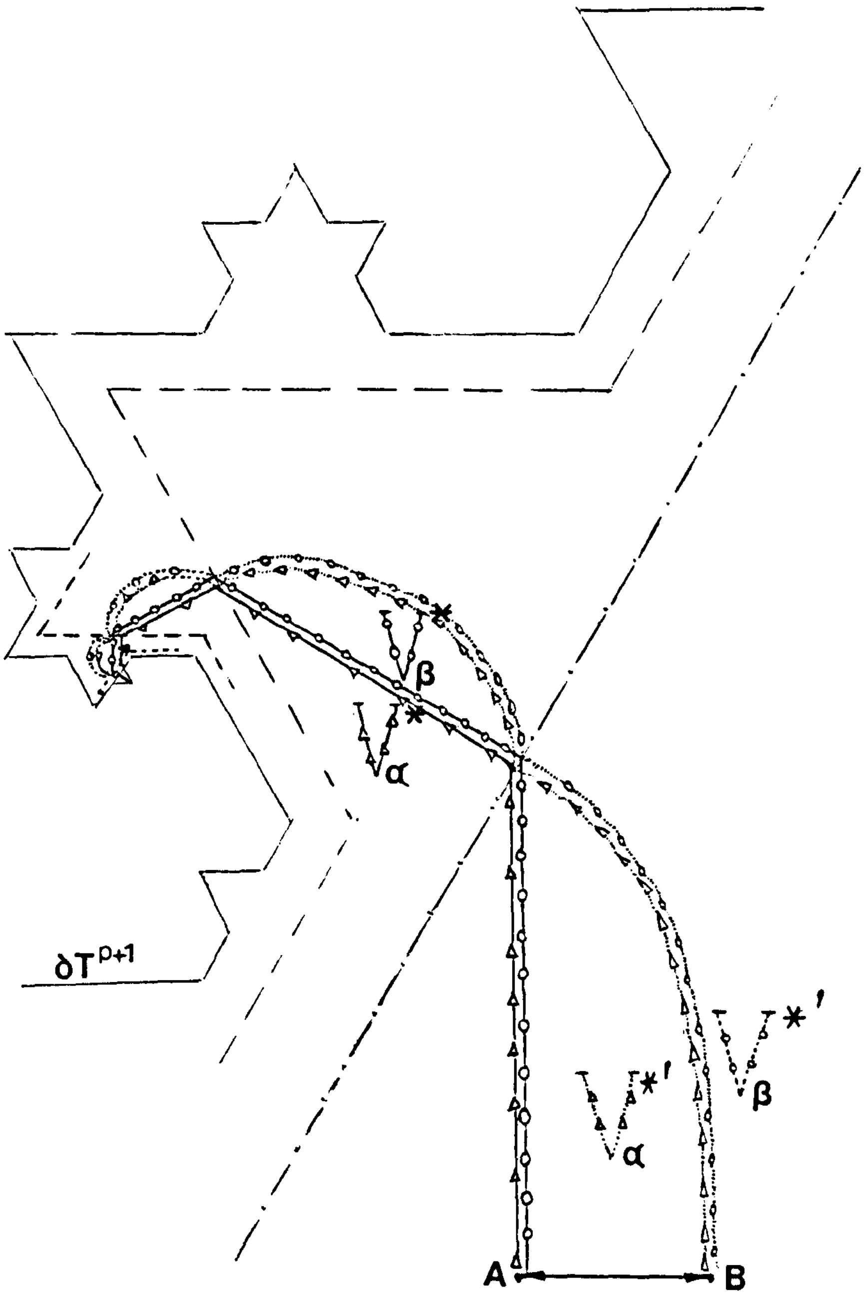
$$\mu^1(\overline{\alpha\beta}) \sim \operatorname{diam} V_\alpha$$

$$\mu^1(\overline{\beta\gamma}) \sim \operatorname{diam} V_\alpha$$

Then

$$\mu^1\left[\mathbb{P}_{s_0}(V'_\alpha)\right] \sim (\operatorname{diam} V_\alpha)^d$$

Figure 8



$$[\mathrm{diam}(\overline{AB})]^d \sim \mu^1(\mathbb{P}_{s_0}(V_\alpha^{*\prime})) \sim \mu^1(\mathbb{P}_{s_0}(V_\beta^{*\prime})) \sim \mu^1(\mathbb{P}_{s_0}(V_\alpha^{*\prime} \cup V_\beta^{*\prime}))$$

But, if V_α and V_β were <u>not</u> close in T^p, then $\mu^1(\mathbb{P}_{s_0}(V_\alpha' \cup V_\beta'))$ can be $\underline{\underline{2}}(\mathrm{diam}(\overline{AB}))^d$.

Figure 9

$$\mu^1[\mathbb{P}_{s_p}\{\sum_{V_\alpha\in\mathbf{A}} V_\alpha(u)\}] \geq$$

$$\mu^1[\mathbb{P}_{s_p}\{\sum_{V_\alpha\in\mathbf{A}_1} V_\alpha\}] \geq \frac{1}{N^2}\mu^1(s_p)$$

...that is, we reduced the general case to the case of $\mathbf{A}_1$.

In the second case, we reduce it to $\mathbf{A}_2$: we set $u' = u \circ M$, and then

$$\mu^1[\mathbb{P}_{s_p}\{\sum_{V_\alpha\in\mathbf{A}} V_\alpha'(u')\}] \geq$$

$$\mu^1[\mathbb{P}_{s_p}\{\sum_{V_\alpha\in\mathbf{A}_2} V_\alpha'(u')\}] \geq$$

$$c\ (1-1/N^2)\,\mu^1(s_p)\,.$$

Claims 1 and 2 imply that we can further restrict ourselves to the case in which

$$\mu^1[\mathbb{P}_{s_p}\{\sum_{\alpha\in A} V_\alpha\}] \sim \mu^1(s_p) \qquad (2)$$

Now we can prove:

Theorem : Let u , h , $V_\alpha{}^h(u)$, be <u>all</u> as described above, with <u>all</u> the restrictions introduced. Then

$$\iint_{u \le h} \operatorname{grad}^2 u \ge c\,h^2 \ .$$

Proof :

Let us introduce first a certain type of transformation of T^p onto T^p . Let us denote:

$$\mathbb{P}_{s_p}\{\sum_{\alpha\in A} V_\alpha\} = \Pi \subset s_p$$

We denote by a_i , $i \in I$, a connected component of $\Pi \subset s_p$ and by b_j a connected component of $s_p - \Pi$.

We write

$$A_i = \{(x, y) \in T^p \mid \mathbb{P}_{s_p}(\{x,y\}) \in a_i \, , i \in I\} \quad ;$$

and B_j is defined in like manner.

We can obtain a transformation of T^p onto T^p by transforming the a_i and the b_j ,--that is, by transforming s_p onto s_p --and by extending this transformation in the obvious way to the corresponding sets A_i and B_j .

Let us go back now to our problem. From equivalence (2) above we know: $\mu^1(s_p) > \mu^1(\Pi) \ge c(\Omega)\, \mu^1(s_p)$.

Notice that $c=c(\Omega)$ can actually be very small.

Suppose, for a moment, that the opposite is true: this constant c is very much near 1 --e.g. $c= 1-1/N^3$ for the Koch snowflake--, let us call it c_1 . Then there is a constant c_2 small enough (e.g. $c_2 = 1/N^3$ for the Koch snowflake) and a close to identity transformation M of T^p onto T^p--fulfilling the hypotheses of the lemma in chapter 3--such that, if we write $u' = u \circ M$, and if we take new local coordinates

$$(\vec{x},\vec{y}) \text{ on } T^p \text{, with the } \vec{x} \text{ axis along } s_p \text{, and } \vec{x} \perp \vec{y},$$

we then have

$\mu^1[\{x \in s_p \mid \exists\, y \text{ such that } u'(x,y)=h\}] = \mu^1(H) \geq c_2\, \mu^1(s_p)$;
where c_1, c_2, and <u>all</u> the constants involved can be made to depend <u>only on Ω</u>.

Then we could write

$$\iint_{u \leq h} \operatorname{grad}^2 u \sim \iint_{u' \leq h} \operatorname{grad}^2 u' \geq \int dx \int u_y'^2\, dy \;,$$

but by our proposition we would have:

$$\left\{ \int u_y'^2\, dy \right\}(x) \geq \left[\frac{h-0}{y(x)}\right]^2 y(x) = \frac{h^2}{y(x)} \geq \frac{h^2}{c\, 1/n^p}$$

for $x \in H$,... and $y(x)$, for that x, is precisely the y referred to in the definition of H --the $y(x)$ such that $u'(x,y(x))=h$.

Then,

$$\int_{x \in H} dx \int u_y'^2\, dy \geq \frac{h^2}{c\, 1/n^p}\, \mu^1(H) \geq \frac{h^2}{c\, 1/n^p}\, C_2\, \mu^1(s_p) = \alpha(\Omega)\, h^2 .$$

It remains, therefore, to transform now u in another function $u'' \in L_1^2(\Omega)$, supp $u'' \subset T^p$, such that the corresponding set $\Pi'' \subset s_p$ <u>does</u> fulfill $\mu^1(\Pi'') \geq c_1(\Omega)\, \mu^1(s_p)$.

In order to do that, we stretch, if necessary, (by means of a transformation of s_p onto s_p) the set Π by a factor $c_3(\Omega)$

which depends on $c_1(\Omega)$ and $c_2(\Omega)$ --by Π'' we denote the stretched set--; and we shrink $s_p - \Pi$ accordingly, by a factor $c_4(\Omega)$ also depending on c_1 and c_2.

We accomplish this by stretching <u>each</u> of those sets a_i by that c_3 factor, and by shrinking <u>each</u> set b_j by the c_4 factor, and we extend this transformation in the obvious way to the corresponding sets A_i and B_j.

It remains only to notice that this transformation M fullfils the hypotheses of the lemma in chapter 3; $u'' = M \circ u$.

Given $u \in L_1^2(\Omega)$, supported on a regular tail T, we will use the last theorem in order to find a locally ruled function

$$\bar{u} \in L_1^2(\Omega)$$

<u>also supported on</u> T, such that,

$$\|\bar{u}\|_{L^2(T)} \geq c_1 \|u\|_{L^2(T)} \quad ;$$

$$\|\mathrm{grad}^2 \bar{u}\|_{L^2(T)} \leq c_2 \|\mathrm{grad}^2 u\|_{L^2(T)} \quad ;$$

that is, we will both increase, e.g., the potential of velocities u, and diminish the energy spent--except for a multiplicative constant--by means of a locally <u>ruled</u> function suported on the smallest tail containing the support of u.

It will also be true:

$$\iint_{\Omega^\varepsilon} u^2 \leq c \iint_{\Omega^\varepsilon} \bar{u}^2$$

and therefore:

$$\bar{Q}^\varepsilon(u) = \frac{\displaystyle\iint_{\Omega^\varepsilon} u^2}{\displaystyle\iint_{\Omega} \mathrm{grad}^2 u} \leq c(\Omega)\, \bar{Q}^\varepsilon(\bar{u})$$

and our theorem on locally ruled functions will ensure

$$\bar{Q}^\varepsilon(u) \leq c(\Omega)\, \varepsilon^{2-d} \quad ,$$

thereby completing our proof of the theorem of Rellich.

It will be enough to find this locally ruled function in the case in which the tail T has <u>size zero</u>.

Let $V^h(u)$, $V_\alpha{}^h(u)$, $\alpha \in A$, denote the same sets as in chapter 6, h is the relocation value. Let us remember now that $[V_\alpha{}^h(u)]'$ was the relocation of $V_\alpha{}^h(u)$ inside T^{p+1}--when $\operatorname{supp} u \subseteq T^p$--; so we will denote with $[V^h(u)]'$ the relocation of $V^h(u)$ inside T^{p+1}.

Theorem :

Given u supported in a tail of size zero: $T = T^0$, then there exist constants $c_i = c_i(\Omega)$, $i \in \{1,2\}$, and a locally ruled function $\bar{u}$ such that,

$$\text{a)} \qquad \iint_{\Omega^\varepsilon} u^2 \le c_1 \iint_{\Omega^\varepsilon} \bar{u}^2$$

$$\text{b)} \qquad \iint_{\Omega} \operatorname{grad}^2 u \ge c_2 \iint_{\Omega} \operatorname{grad}^2 \bar{u} \quad .$$

Proof :

We will construct the ruled function in stages. We write:

$$T = (T - T^1) + (T^1 - T^2) + \ldots$$

and we will construct $\bar{u}$ first in $T - T^1$, next in $T^1 - T^2$, ... etc; in an iterative way: the procedure for the construction of $\bar{u}$ in $T^i - T^{i+1}$ will be the same as in $T^{i+1} - T^{i+2}$.

We want to apply the theorem in chapter 6 tolhe case $p = 0$, $p+1 = 1$; T^p and T^{p+1} of the theorem will be our T and $T^1 \subset T$. By h_1 we denote the $0 \to 1$-relocation value.

We want to ensure $h_1 > 0$, so we slightly modify the function u:

Let $f = f(x,y)$ be such that:

$$f \in L_1^2(\Omega) \ ; \ \operatorname{supp} f = T \ ;$$

f is linear in T, and its gradient is arbitrarily small when compared with the one of u, say

$$\iint_T \operatorname{grad}^2 f \le 0.01 \iint_T \operatorname{grad}^2 u \quad ;$$

in other words, f is very small, but different from zero on T.

By defining $u^* = \sup\{u, f\}$, we obtain a function as near u as we may want, but whose support is strictly T.

We have:

$$\iint_{\Omega^\varepsilon} u^2 \le \iint_{\Omega^\varepsilon} (u^*)^2 \quad ;$$

$$\iint_\Omega \operatorname{grad}^2 u^* = \iint_{u \ge f} + \iint_{u<f} = \iint_{u \ge f} \operatorname{grad}^2 u + \iint_{u<f} \operatorname{grad}^2 f \le$$

$$\iint_T \operatorname{grad}^2 u + \iint_T \operatorname{grad}^2 f \le 1.01 \iint_T \operatorname{grad}^2 u \quad .$$

Therefore,

$$\overline{Q}^\varepsilon(u) \le 1.01\, \overline{Q}^\varepsilon(u^*) \quad ;$$

and both

$$\overline{Q}^{\varepsilon}(u^*) \text{ and } \overline{Q}^{\varepsilon}(u)$$

can be made to be arbitrarily close. We will work with u^* instead of u, and for the sake of simplicity we will continue calling it u.

Now h_1 is strictly positive.

This procedure of (arbitrarily) slight modification of a function will be used in each iteration of the process of contruction of $\bar{u}$.

We need one more notation:
On a set $V_\alpha^{h_1}(u)$, $\alpha \in A$, we have:

$$u = h_1 + \Delta_\alpha \ ; \ \text{supp}\, \Delta_\alpha \subseteq V_\alpha^{h_1}(u) \ ;$$

when relocating each $V_\alpha^{h_1}$ inside T^1 we will denote by Δ'_α the corresponding relocated function:

$$\text{supp}\, \Delta'_\alpha \subseteq [V_\alpha^{h_1}]' \ .$$

Let us now consider

$$u_1 = \begin{cases} h_1 \text{ in } T - T^1 \\ h_1 \text{ in } T^1 - [V^{h_1}(u)]' \\ h_1 + \Delta'_\alpha \text{ in each } [V_\alpha^{h_1}(u)]' \end{cases}$$

Then, using the theorem in the last chapter, we have:

$$\overline{Q}^{\varepsilon}(u) \le \frac{\displaystyle\iint_{T^{\varepsilon}} u_1^2}{\displaystyle\iint_{0 \le u \le h_1} \text{grad}^2 u + \iint_{u \ge h_1} \text{grad}^2 u} \le$$

$$\frac{\iint_{T^\varepsilon} u_1^2}{c\,h_1^2 + \iint_{T^1} \operatorname{grad}^2 u_1} = \frac{\iint_{T^\varepsilon - (T^1)^\varepsilon} h_1^2 + \iint_{(T^1)^\varepsilon} u_1^2}{c\,h_1^2 + \iint_{T^1} \operatorname{grad}^2 u_1} \quad .$$

We focuss next on

$$\iint_{(T^1)^\varepsilon} u_1^2 \quad \text{and} \quad \iint_{T^1} \operatorname{grad}^2 u_1$$

in the same way in which we focussed on

$$\iint_{T^\varepsilon} u^2 \quad \text{and} \quad \iint_{T} \operatorname{grad}^2 u \quad ,$$

and we will iterate the method from the case $T - T^1$ & T^1 to the case $T^1 - T^2$ & T^2: We have

$u_1 \geq h_1$ in T^1 instead of $u \geq 0$ in T ; and

$u_1\big|_{\partial T^1 \cap \operatorname{Int} T} = h_1$ instead of $u\big|_{\partial T \cap \operatorname{Int} \Omega} = 0$,

so h_1 will take for T^1 the place that the value 0 had for T: we will work from h_1 "upwards".

For $p = 1$, $p + 1 = 2$, and u_1 in T^1 we will find the $p \to (p+1)$-relocation value h_2 of the theorem in chapter 6. Again we want to make that slight modification to u_1 that will yield $h_2 > 0$. When we did it before for T and u it was by means of a slight increase of u --denoted by u^* -- , we used a linear function $f = f_0$ of constant gradient, such that

$$\mathrm{grad}^2 f \leq \frac{0.01}{\mu^2(T)} \iint_T \mathrm{grad}^2 u \quad ;$$

so that

$$\iint_T \mathrm{grad}^2 u^* \leq \iint_T \mathrm{grad}^2 u + \iint_T \mathrm{grad}^2 f \leq$$

$$\iint_T \mathrm{grad}^2 u + 0.01 \iint_T \mathrm{grad}^2 u \quad .$$

We want now, likewise, a linear function f_1 on T^1 of small gradient, such that

a) $\displaystyle\iint_{T^1} \mathrm{grad}^2 f_1 \leq 0.1\, c\, h_1^2$, b) $\displaystyle\mathrm{grad}^2 f_1 \leq \frac{0.001}{\mu^2(T)} \iint_T \mathrm{grad}^2 u$

and we call u_1^* the correspondingly modified function.

Condition b) for f_1, the similar one for $f = f_0$ and similar ones for f_2, f_3, ... to be defined in succesive iterations, will imply that the sum of **all** the slight modifications introduced will be very small.

Condition a) means that we can now iterate what we have already done:

$$\overline{Q}^\varepsilon \leq \frac{\displaystyle\iint_{T^\varepsilon-(T^1)^\varepsilon} h_1^2 + \iint_{(T^1)^\varepsilon} u_1^2}{\displaystyle 0.9\, c\, h_1^2 + 0.1\, c\, h_1^2 + \iint_{T^1} \mathrm{grad}^2 u_1} \leq$$

$$\frac{\displaystyle\iint\limits_{T^{\varepsilon}-[T^1]^{\varepsilon}} h_1^2 + \iint\limits_{[T^1]^{\varepsilon}} [u_1^*]^2}{\displaystyle 0.9\,c\,h_1^2 + \iint\limits_{T^1} \mathrm{grad}^2\, u_1^*} \leq$$

$$\frac{\displaystyle\iint\limits_{T^{\varepsilon}-[T^1]^{\varepsilon}} h_1^2 + \iint\limits_{[T^1]^{\varepsilon}-[T^2]^{\varepsilon}} (h_1+h_2)^2 + \iint\limits_{[T^2]^{\varepsilon}} u_2^2}{\displaystyle 0.9\,c\,h_1^2 + c\,h_2^2 + \iint\limits_{T^2} \mathrm{grad}^2\, u_2} \leq$$

$$\frac{\displaystyle\iint\limits_{T^{\varepsilon}-[T^1]^{\varepsilon}} h_1^2 + \iint\limits_{[T^1]^{\varepsilon}-[T^2]^{\varepsilon}} (h_1+h_2)^2 + \iint\limits_{[T^2]^{\varepsilon}} [u_2^*]^2}{\displaystyle 0.9\,c\,h_1^2 + 0.9\,c\,h_2^2 + \iint\limits_{T^2} \mathrm{grad}^2\, u_2^*}$$

and so on, where u_2, u_2^*, h_2, ..., etc. are the analogues of u_1, u_1^*, and h_1.

The numerator thus obtained, when iterating this process, is

$$\iint\limits_{\Omega^{\varepsilon}} u^2 \;;$$

where the discontinuous function u equals h_1 in $T-T^1$, h_1+h_2 in T^1-T^2, ... and so on.

Now, if we modify this function u in order to make it continuous in the jump from $T^{i-1}-T^i$ to T^i-T^{i+1}, $i \in \mathbb{N}$, by making u go linearly from

$$\sum_{j=1}^{i} h_j \quad \text{to} \quad \sum_{j=1}^{i+1} h_j$$

along a narrow strip $\delta_i \subset T^i$, as shown in figure (1); and if we do it by keeping the same shape and proportion of δ_i in relation to T^i, ... then we have a function $\bar{u}$, locally ruled on all Ω, supported on T, such that

a) Our numerator fulfills:

$$\iint_{\Omega^\varepsilon} u^2 \leq c \iint_{\Omega^\varepsilon} \bar{u}^2$$

where $c = c(\Omega) = c(\delta_i)$. Therefore

$$\iint_{\Omega^\varepsilon} u^2 \leq c \iint_{\Omega^\varepsilon} \bar{u}^2 \quad .$$

b) The corresponding denominator fulfills:

$$\sum_{i \in \mathbb{N}} 0.9\, c\, h_i^2 = c(\Omega) \iint_T \operatorname{grad}^2 \bar{u} \quad ,$$

and therefore

$$\bar{Q}^\varepsilon(u) \leq c(\Omega)\, \bar{Q}^\varepsilon(\bar{u}) \quad .$$

Notice also: the fact that the sum of all the perturbations introduced is very small when compared to

$$\iint_T \operatorname{grad}^2 u$$

implies that

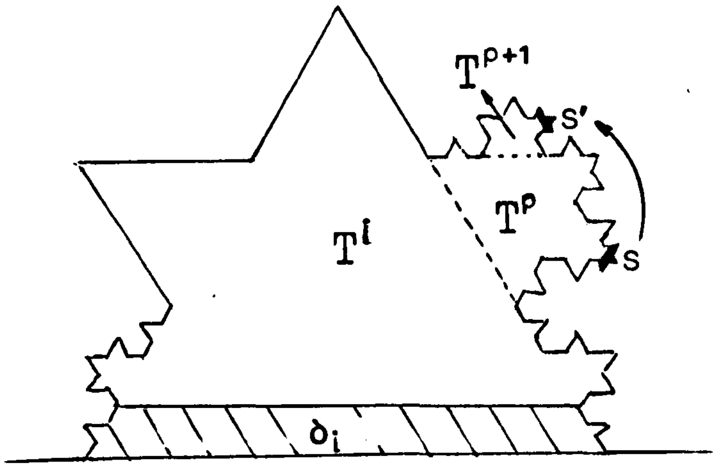


Figure 1

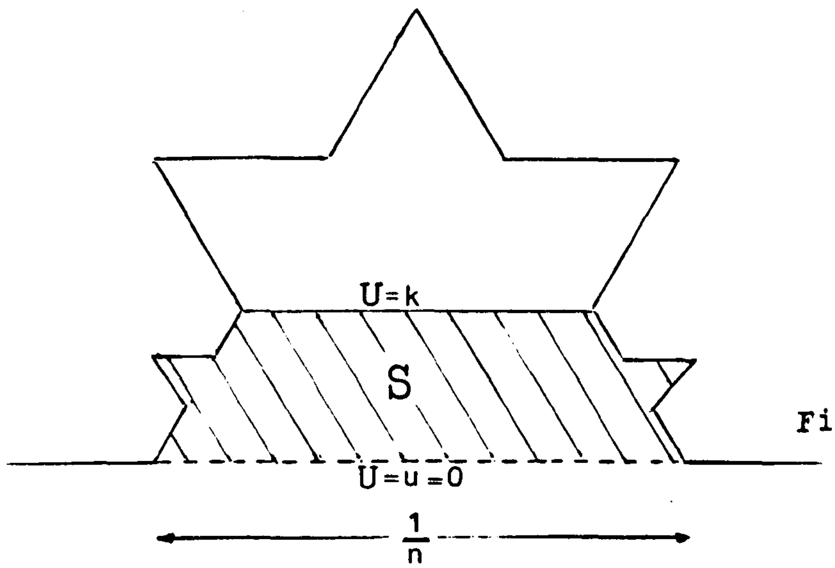


Figure 2

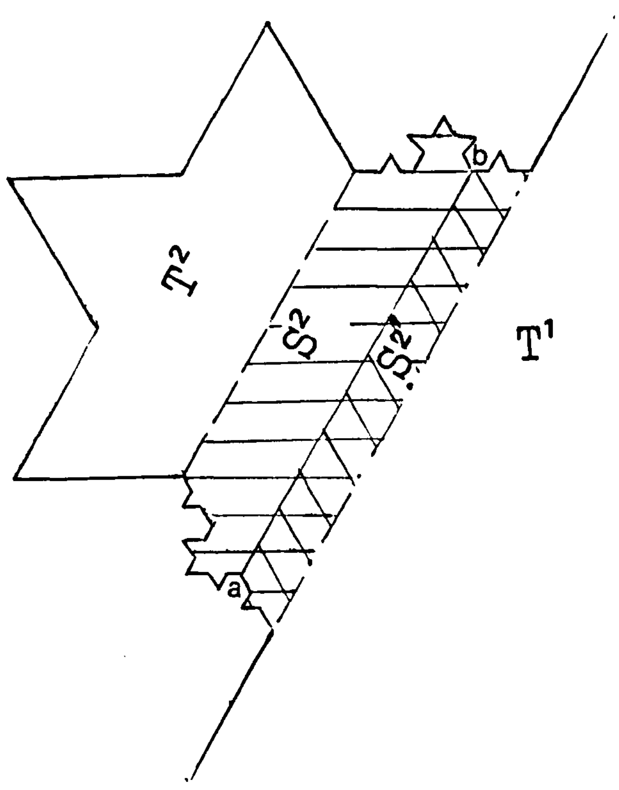


Figure 3

$$\iint_T \operatorname{grad}^2 \bar{u} = c_1 \sum_{i \in N} h_i^2 \le c_2 \iint_T \operatorname{grad}^2 u \quad .$$

Appendix to Chapter 7

In this section we will give another solution to the problem of chapter 7: given u as before, supported on a tail T ; we want a function $\tilde{u}$, locally linear, also supported in T , such that, except for a constant $c=c(\Omega)$, the L^2 norm of $\tilde{u}$ is bigger than the L^2 norm of u , while the L^2 norm of the gradient of $\tilde{u}$ is smaller than the corresponding one of u .

The difference between this new method and the one just exhibited is twofold: on the one hand, the $\tilde{u}$ constructed below can have L^2 norm <u>much</u> bigger than the norm of the previous $\tilde{u}$ in chapter 7, while the L^2 norm of its gradient can be <u>much</u> smaller than the corresponding one of $\tilde{u}$ in chapter 7.

But, on the other hand, from this new method, an estimation of $Q^{\varepsilon}(u)$ is not immediately obtained as in the chapter.

We need some more notation:

Let $\mathbf{R}$ be any bounded planar region; let us denote:

$\mathbf{R}_\lambda = \{ (x',y') \mid (x',y') = \lambda\,(x,y) \,;\, (x,y) \in \mathbf{R} \}$; $\lambda \geq 1$.

and let us call $\mathbf{R}_\lambda$ a <u>λ-expansion of $\mathbf{R}$</u> .

For instance: a tail T^p of size p is an n-expansion of any T^{p+1} ; any two tails are (up to a congruence) one an expansion of the other, for some $\lambda = n^k$, $k \in \mathbb{N}$.

Next, let $\mathbf{R}$ be as above, to any $u \in L_1^2(\mathbf{R})$ we associate a function $u' \in L_1^2(\mathbf{R}_\lambda)$, such that $u'(x',y') = u'(\lambda x, \lambda y) = u(x,y)$;

and call u' the <u>function associated to u and to $\mathbb{R}_\lambda$</u> .

The next two lemmas need no proof:

Lemma 1 : Let $\mathbb{R}$, $\mathbb{R}_\lambda$, u , u' , and λ be as above. Then

$$\iint_{\mathbb{R}} \text{grad}^2 u \, dx \, dy = \iint_{\mathbb{R}_\lambda} \text{grad}^2 u' \, dx' \, dy'$$

Lemma 2 : Under the same hypotheses as Lemma 1 we have:

$$\iint_{\mathbb{R}_\lambda} (u')^2 \, dx' \, dy' = \lambda^2 \iint_{\mathbb{R}} u^2 \, dx \, dy$$

We need also:

Lemma 3 : Let $u_1 , \ldots , u_n$ belong to $L_1^2(\Omega)$; $u^2 = u_1^2 + \ldots + u_n^2$. Then
$\text{grad}^2 u \leq \text{grad}^2 u_1 + \ldots + \text{grad}^2 u_n$.

Proof : One application of Schwartz gives:

$$u_x = \frac{1}{2\sqrt{u_1^2 + \ldots + u_n^2}} (2u_1 u_{1_x} + \ldots + 2u_n u_{n_x}) \leq$$

$$\frac{\sqrt{u_1^2 + \ldots + u_n^2} \, \sqrt{u_{1_x}^2 + \ldots + u_{n_x}^2}}{\sqrt{u_1^2 + \ldots + u_n^2}} = \sqrt{u_{1_x}^2 + \ldots + u_{n_x}^2}$$

Therefore

$$u_x^2 \leq u_{1_x}^2 + \ldots + u_{n_x}^2$$

and therefore

$$u_x^2 + u_y^2 \leq \mathrm{grad}^2 u_1 + \ldots + \mathrm{grad}^2 u_n$$

Note : Notice that, if

$$\sum_{i=1}^{\infty} \mathrm{grad}^2 u_i \quad \text{and} \quad u^2 = \sum_{i=1}^{\infty} u_i^2$$

are both integrable, then we can extend the result for finite sums in Lemma 3 to the corresponding result for an infinite sum.

We still need one more notation:

Let a region R be a non rampant sum of two regions: $R = R_1 + R_2$. Let $A \subset R_1$. We say that the set $A \subset R_1$ is <u>mappable out of</u> R_1 and <u>into</u> R_2, if there is a set $A' \subset R_2$; A' congruent to A. The set A' will be called <u>the mapping out of</u> A.

Example : In the theorem that follows, we will have $R=T^p$, for some $p \geq 1$; and R_1+R_2 as follows:

$$T^p = (T^p - T^{p+1}) + T^{p+1} \; ; \; T^{p+1} \subset T^p .$$

Then any set S contained in any tail

$$T^{p+k+1} \subset T^p - T^{p+1} \; ; \; k \in \mathbb{N},$$

is mappable out of $T^p - T^{p+1}$ and into T^{p+1} with the property:

$$\partial S \cap \partial T^p \text{ and } \partial S' \cap \partial T^{p+1} \text{ are congruent} \qquad (1)$$

(see figure (1)).

In other words:

The fractal regularity of the (n,N) process implies that the boundaries of T^p and T^{p+1} have, locally, the same shape.

In the remainder of this section all our mappings out of sets S will have the property (1).

Finally, let $u \in L_1^2(T)$; $\operatorname{supp} u = S \subset T^{p+1+k} \subset T^p - T^{p+1}$, for some $k \in \mathbb{N}$.

Let S' be the mapping out of S , out of $T^p - T^{p+1}$, and into T^{p+1} . Let the function u' be defined on S' such that, if the point $(x,y) \in S$ is mapped into $(x',y') \in S'$, then $u'(x',y') = u(x,y)$.

We will say that u is mapped out of $T^p - T^{p+1}$ into T^{p+1} , u' being the mapped out function.

Theorem :

Let $u \in L_1^2(T)$; $T = T^1$.

Let us write: $T = (T^1 - T^2) + T^2$ for some $T^2 \subset T^1 = T$.

Then there exists $\mathbf{u} \in L_1^2(T^2)$, locally ruled in $T^1 - T^2$, such that

$$\iint_\Omega u'^2 \le c \iint_{T^1-T^2} \mathbf{u}^2 + \iint_{T^2} \mathbf{u}^2 \quad ;$$

$$\iint_\Omega \operatorname{grad}^2 u \ge \alpha(\Omega) \iint_{T^1-T^2} \operatorname{grad}^2 \mathbf{u} + \iint_{T^2} \operatorname{grad}^2 u \quad ;$$

and $\mathbf{u}$ has constant value k in $\partial(T^1 - T^2) \cap \partial T^2$.

Comment : The method can be iterated in order to find $\mathbf{u}_2$ on T^2 ; $\mathbf{u}_2 = k = k_1$ in $\partial(T^1 - T^2) \cap \partial T^2$;

$$u_2 \text{ locally ruled in } T^2\text{-}T^3 \;;\; T^3 \subset T^2$$
$$\text{and } u_2 = k_2 \text{ in } \partial(T^2\text{-}T^3) \cap \partial T^3 \text{ , such that:}$$

$$\iint_{T^2} u_1^2 \leq c \iint_{T^2-T^3} u_1^2 + \iint_{T^3} u_2^2 \;\;;$$

where we wrote $u=u_1$; and

$$\iint_{T^2} \operatorname{grad}^2 u_1 \geq c \iint_{T^2-T^3} \operatorname{grad}^2 u_2 + \iint_{T^3} \operatorname{grad}^2 u_2 \;\;;$$

where the constants are the same as those in the first stage; and

$$u_2 \in L_1^2(T^3) \;.$$

Further iterations of this method allow us to define a function U supported, like u , on T , locally ruled on all

$$(T^1\text{-}T^2) + (T^2\text{-}T^3) + \ldots = T$$

by writing $U = u_i$ on $T^i\text{-}T^{i+1}$, $i \in \mathbb{N}$; U will fulfill:

$$\iint_{\Omega} u^2 \leq c \iint_{\Omega} U^2 \;\;;\; \text{and} \;\; \iint_{\Omega} \operatorname{grad}^2 u \geq c \iint_{\Omega} \operatorname{grad}^2 U \;,$$

and it will be seen that

$$\iint_{\Omega} U^2 \text{ can be \underline{much} larger than } \iint_{\Omega} u^2 \;,$$

and also

$$\iint_{\Omega} \operatorname{grad}^2 U \;\text{ much smaller than } \iint_{\Omega} \operatorname{grad}^2 u \;.$$

Proof : It will be noticed that the case of any regular fractal in not different from the one of the Koch snowflake, regarding the construction that follows. Let us focus, then, on that particular case.

The locally ruled function U , in $T-T^2 = T^1-T^2$ will be, just like the ruled function constructed in chapter 7, linearly going from zero to a height $k \in \mathbf{R}^+$, along a strip $S=S^1 \subset T^1-T^2$, as shown in figure (2) for the Koch snowflake, and of constant value k on $(T^1-T^2) - S^1$.

In T^2-T^3 ; $T^3 \subset T^2$, we consider the strip S^2 analogous to S^1 ; and also a strip $S^{2'} \subset S^2$, its width being $1/n$ of the width of S^2 , as seen in figure (3).

We define now the value of the constant $k \in \mathbf{R}^+$ just referred to above: Proceeding in an analogous way to the proof of chapter 7, we write:

$$k=\inf\{k' \in \mathbf{R}^+ \mid \sup_{\alpha \in A} \operatorname{diam}(V_\alpha^{k'}(u)) \leq 1/n^4 \,; V_\alpha^{k'}(u) \subset S^{2'}+(T^1-T^2) \ \ \forall \alpha \in A\}$$

Applying the corollary of the lemma in 6th chapter, we know that:

$$\iint_{\{(T^1-T^2)+S^{2'}\}-\sum_{\alpha \in A} V_\alpha^k(u)} \operatorname{grad}^2 u \geq \alpha(\Omega)\, k^2 \qquad (1)$$

Let us consider the set T^2 as open, and the sets $V_\alpha{}^k(u)$, $\alpha \in A$, as closed. Let us write $\alpha_1 \in A_1 \subset A$ when

$$\partial V_{\alpha_1}^k(u) \cap \overline{\partial S^{2'} \cap T^2} \neq \emptyset \ ; \ A_2 = A - A_1 \ ;$$

in other words, A_1 consists of the indices α_1 for which $\partial V_{\alpha_1}{}^k(u)$ has one point in common with the segment $[a,b]$ shown in figure (3).

Let us consider the auxiliary ruled function u^r as going from zero to k linearly along the strip S^1, and being constant in T^1-S^1

If $A_2 = \emptyset$, then let us write:

$$u = \begin{cases} u^r \text{ in } T^1 - T^2 \\ \sup\{k,u\} \text{ in } T^2 \end{cases} \qquad (2)$$

If A_2 is not empty, consider all $V_\alpha^k(u)$, $\alpha \in A_2$, and the associated functions $\Delta_\alpha = u(x,y) - k$; $(x,y) \in V_\alpha^k(u)$

$$\operatorname{supp} \Delta_\alpha \subset V_\alpha^k(u) \ ; \ \alpha \in A_2 \ .$$

For brevity sake we will denote $V_\alpha^k(u) = V_\alpha$ when there is no possibility of confusion.

Let us perform now an n-expansion of each V_α, $\alpha \in A_2$, and let Δ'_α be the corresponding function associated both to V_α and to the n-expansion of V_α. Each such V_α expanded is mappable into T^2-$S^{2'}$.

The corresponding mapped out of each Δ'_α, $\alpha \in A_2$, will--for the sake of brevity--also be called Δ'_α; each such Δ'_α is understood now as being supported in T^2-$S^{2'}$.

And now we write:

$$u = \begin{cases} u^r \text{ in } \{(T^1 - T^2) + S^{2'}\} - \sum_{\alpha \in A_1} V_\alpha \\ \sqrt{\sum_{\alpha \in A_2} (\Delta'_\alpha)^2 + (\sup\{k,u\})^2} \quad \text{in the rest of } T^2 \end{cases} \qquad (3)$$

Notice that

$$\iint_{T^2} \sum_{\alpha\in A_2} (\Delta'_\alpha)^2 < n^2 \| u \|^2_{L^2}$$

Notice also that (2) is a particular case of (3), therefore, let us consider the definition of u given in (3).

For brevity we will write

$$(T^1\text{-}T^2) + S^{2'} - \sum_{\alpha\in A_1} V_\alpha = A \ ; \ T - A = B$$

A is sligtly larger than $T^1\text{-}T^2$, B is slightly smaller than T^2. It will be enough to prove the theorem for A and B instead of $T^1\text{-}T^2$ and T^2: notice that this will pose no problem for the iteration of the construction, the function u we are constructing may possibly stick at constant value k for a subset of the small strip $S^{2'}$.

(1) and (3) can be rewritten:

$$\iint_{A - \sum_{\alpha\in A_2} V_\alpha} \mathrm{grad}^2 u \geq \alpha(\Omega)\, k^2 \qquad (1')$$

$$u = \begin{cases} u' \text{ in } A \\ \sqrt{\sum_{\alpha\in A_2} (\Delta'_\alpha)^2 + (\sup\{k,u\})^2} \ \text{ in } B \end{cases} \qquad (3')$$

and each Δ'_α is now supported in B.

Lemma 2 implies:

$$\iint_{\Omega} u^2 = \iint_{T^1} u^2 = \iint_{A-\sum_{\alpha\in A_2} V_\alpha} u^2 + \sum_{\alpha\in A_2} \iint_{V_\alpha} u^2 + \iint_{B} u^2 \leq$$

$$k^2 \mu^2(A-\sum_{\alpha\in A_2} V_\alpha) + \sum_{\alpha\in A_2} \iint_{V_\alpha} (k+\Delta_\alpha)^2 + \iint_{B} \sup^2\{k,u\} \leq$$

$$k^2 \mu^2(A-\sum_{\alpha\in A_2} V_\alpha) + \sum_{\alpha\in A_2} \{2k^2\mu^2(V_\alpha) + 2\iint_{V_\alpha} \Delta_\alpha^2\} + \iint_{B} \sup^2\{k,u\} \leq$$

$$2k^2\mu^2(A) + n^2 \sum_{\alpha\in A_2} \iint_{V_\alpha} \Delta_\alpha^2 + \iint_{B} \sup^2\{k,u\} \leq$$

$$c(\Omega) \iint_{A} (u^r)^2 + \sum_{\alpha\in A_2} \iint_{B} (\Delta'_\alpha)^2 + \iint_{B} \sup^2\{k,u\} =$$

$$c(\Omega) \iint_{A} u^2 + \iint_{B} u^2 \quad .$$

Using now (1') and lemma 1 we have (the constant $c(\Omega)$ chanching from line to line, but always denoted $c(\Omega)$ for brevity):

$$\iint_{\Omega} \operatorname{grad}^2 u = \iint_{T^1} \operatorname{grad}^2 u \geq \iint_{A-\sum_{\alpha\in A_2} V_\alpha} \operatorname{grad}^2 u + \sum_{\alpha\in A_2} \iint_{V_\alpha} \operatorname{grad}^2 u + \iint_{B} \operatorname{grad}^2 \sup\{k,u\}$$

$$\geq \alpha(\Omega)k^2 + \sum_{\alpha \in A_2} \iint_{V_\alpha} \operatorname{grad}^2 \Delta_\alpha + \iint_B \operatorname{grad}^2 \sup\{k,u\} =$$

$$\alpha(\Omega) \iint_A \operatorname{grad}^2 u' + \sum_{\alpha \in A_2} \iint_B \operatorname{grad}^2 \Delta'_\alpha + \iint_B \operatorname{grad}^2 \sup\{k,u\} \geq$$

$$\alpha(\Omega) \iint_A \operatorname{grad}^2 \mathbf{u} + \iint_B \operatorname{grad}^2 \sqrt{\sum_{\alpha \in A_2} (\Delta'_\alpha)^2 + \sup^2\{k,u\}} =$$

$$\alpha(\Omega) \iint_A \operatorname{grad}^2 \mathbf{u} + \iint_B \operatorname{grad}^2 \mathbf{u} \quad ,$$

we used the lemma 3 at the end.

Let us finish this appendix with an example of how this new method can work. When following the first method, a bounded function u will be inevitably replaced by a locally linear function $\tilde{u}$ having the very same bound. The gradient of u can be very large, yet the height of $\tilde{u}$ will not reflect this. This new method, on the contrary, can exploit the magnitude of the gradient and "transform" it into height.

We will construct now a positive function u supported in the Koch snowflake, bounded by $1+\delta$, $\delta<<1$. We will remember that the Koch snowflake can be written as a non rampant sum of triangles A_i^p of different sizes. The function u will be supported in the triangle A^0 . The sides of A^0 have length 1 .

We need first a construction:
We consider another triangle $A_0'\subset A^0$, centered on the barycentre of A^0 and with sides parallel to the sides of A^0 . The length of the side of A_0' is $1-\delta/n$, where

$$\delta = \frac{1}{10^6\ 1/(1-1/n)} \quad .$$

Next we divide A_0' into $9=3^2=n^2$ equal triangles A_1^i , whose sides will have length equal to

$$\frac{1-\delta/n}{n} = 1/n - \delta/n^2 \ .$$

In a way similar to the one used in order to obtain A_0' , we obtain $A_1^{i'} \subset A_1^i$, centered like A_1^i, with sides parallel to those of A_1^i, the length of each side equal to $1/n - \delta/n^2 - \delta/n^3$ ...

Next, we will divide each triangle $A_1^{i'}$ into n^2 equal triangles A_2^i whose sides will have length equal to $1/n^2 - \delta/n^3 - \delta/n^4$, ... and obtain a total of n^4 triangles $A_2^{i'} \subset A_2^i$, with sides of length $1/n^2 - \delta/n^3 - \delta/n^4 - \delta/n^5 = 1/n^2 - \delta/n^3(1 + 1/n + 1/n^2)$ ... In the $p^{\underline{th}}$ stage the corresponding value will be $1/n^p - \delta/n^{p+1}(1 + 1/n + ... + 1/n^p)$. The n^p-expansion of any of these triangles $A_p^{i'}$ will have sides with length $1 - \delta/n(1 + 1/n + ... + 1/n^p) = 1 - \delta/n - \delta(1/n^2 + ... + 1/n^p)$, and therefore it can be relocated inside A_0'.

We will construct now our function u in stages. Supp u will be A^0. All the functions constructed below will be in $L_1^2(\Omega)$

Let u_0 be such that:

$$\text{supp}\, u_0 = A^0 \;;\; u_0 = 1+\delta \text{ in } A_0' .$$

The function u_1 is a modification of u_0:

$\text{supp}\, u_1 = A^0$; $u_1 = 1+\delta$ in each $A_1^{i'}$ for every $i \in \{1, ..., n^2\}$ and also:

a) $z<\delta/2$ implies: there is only one connected component $V^z(u_1)$; and $A_0' \subset V^z(u_1) \subset A^0$;

b) $A_1^{i'} \subset V_i^{\delta/2}(u_1) \subset A_1^i$ for every $i \in \{1, ..., n^2\}$, and

c) the n-expansion of each $V_i^{\delta/2}(u_1)$ can be contained in A_0'.

Let us iterate this construction just once more:

u_2 is a modification of u_1:

$\mathrm{supp}\, u_2 = A^0$; $u_2 = 1+\delta$ in each $A_2^{i'}$ for every $i \in \{1, \ldots, n^4\}$

and also:

a) $z<\delta/2$ implies: there is only one connected component $V^z(u_2)$; and $A_0' \subset V^z(u_2) \subset A^0$;

b) $\delta/2 \leq z < \delta/2 + \delta/2^2$ implies: there are exactly n^2 connected components $V_i^z(u_2)$, $A_1^{i'} \subset V_i^z(u_2) \subset A_1^i$, and their n-expansions can be contained in A_0' ;

c) $A_2^{i'} \subset V_i^{\delta/2+\delta/2^2}(u_2) \subset A_2^i$ for $i \in \{1, \ldots, n^4\}$, and

d) each $V_i^{\delta/2+\delta/2^2}(u_2)$, when n^2-expanded, can be contained in A_0' .

Let us stop this process at the stage p , so $u=u_p$. Notice that: when intersecting the graph of u with planes $z=\delta/2=z_1$, $z=\delta/2+\delta/2^2=z_2$, ... $z=\delta/2+\delta/2^2+\ldots+\delta/2^i=z_i$ ..., we can either relocate the corresponding $V_j^{z_i}(u)$ in smaller tails, n-expand-and-relocate them in smaller tails, n^2-expand-and-relocate in smaller tails, ... as the case may be, depending on the height z_i and on the size of the smaller tails, ... so we can finally set ourselves to construct our locally linear function $\bar{u}$ ...

However, we want to give here a <u>very</u> simple example of our method: we will simply consider the plane $z = \delta/2 + \delta/2^2 + \ldots + \delta/2^p = z_p$ and find the corresponding sets $V_i^{z_p}(u) = V_i^{z_p}(u_p)$, and the corresponding associated functions Δ_i with support in each $V_i^{z_p}(u)$, $i \in \{1, \ldots, n^{2p}\}$.

Each $V_i^{z_p}(u)$, when n^p-expanded, will be contained in A_0'

and in the corresponding n^p-expansion of $A_p^{i'} \subset V_i^{z_p}(u)$ we have:

$$\Delta_i' = 1 + \delta - (\delta/2 + \ldots + \delta/2^p) > 1 .$$

Consider now the function

$$u' = \sqrt{\sum_{i=1}^{n^{2p}} \Delta_i'^2} \quad ;$$

we know that:

$$\iint \mathrm{grad}^2 u' < \iint \mathrm{grad}^2 u \quad , \text{and}$$

in the n^p-expansion of $A_p^{i'}$ we have:

$$u' > \sqrt{\sum_{i=1}^{n^{2p}} 1^2} = \sqrt{n^{2p}} = n^p \quad ,$$

with p arbitrarily large ...

Chapter 8

In this section, we will deal with the pathological case in which the dimension d is 2; and the $(n,N) = (n,n^2)$ process of replacement is still <u>regular</u>. The value $d=2$ means that the limiting set $\partial\Omega$ obtained in this case is not the boundary of a region Ω; we will denote it by R, $\mu^2(R) \neq 0$.

We will assume also $\mathrm{Int}\, R \neq \emptyset$; and we write:

$$\mathring{\Omega} = \liminf \mathrm{Int}(\Omega^p) \ ;$$

i.e. the set of points x belonging to every $\mathrm{Int}\,\Omega^p$ for all $p>p_x$, for some $p_x \in \mathbb{N}$.

$$\mathring{\Omega} \text{ and } R$$

are non rampant; their non rampant sum will be denoted by Ω, i.e.

$$\Omega = R + \mathring{\Omega}$$

see figure (1) for an example.

We will show now that the pathological case $d=2$ cannot be considered as resulting from a <u>regular</u> process of replacement in which $\Omega^p \subset \Omega^{p+1}$ for every $p \in \mathbb{N}$; and in which Ω is bounded.

For let us suppose that this is the case:

In the (n,N) process of replacement that gives Ω^p from Ω^{p-1}, let us focus on only <u>one</u> segment s in the polygon $\partial\Omega^{p-1}$; $\mu^1(s) = 1/n^{p-1}$, and let us denote by i_p that subset of $\Omega^p - \Omega^{p-1}$ whose boundary intersects the segment s.

Now, there exists a constant $c=c(\Omega)$ such that

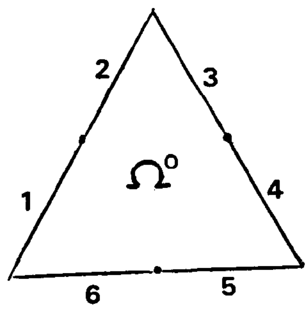


Fig. 1(a)

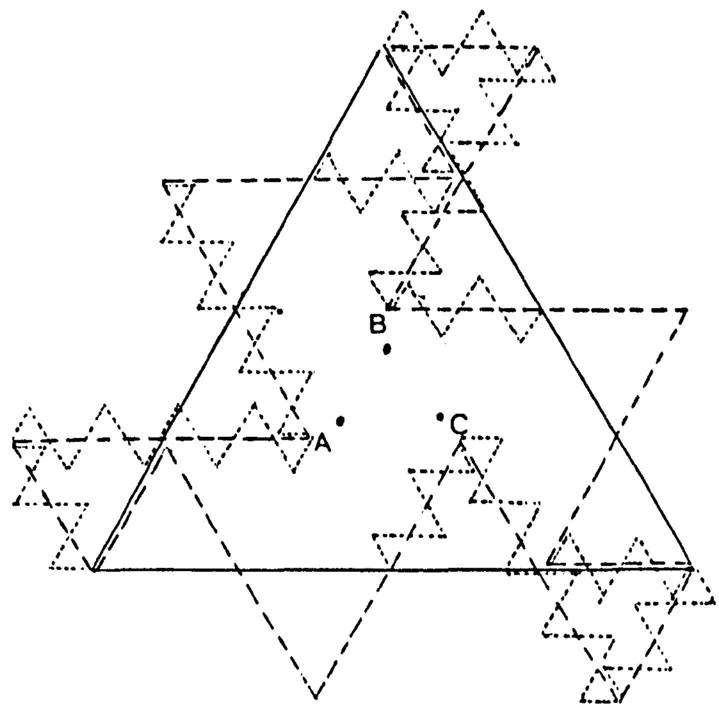


Fig. 1(b)

Figure 1

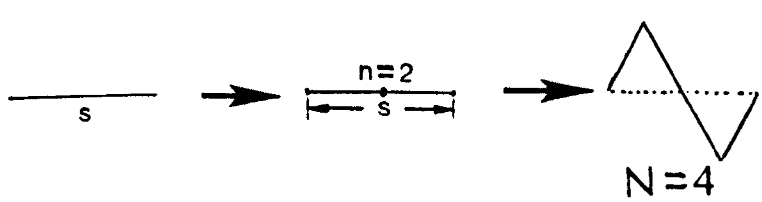


Fig. 2 (a)

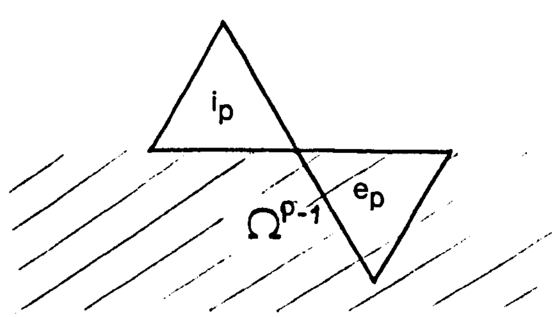


Fig. 2 (b)

Figure 2

$$\mu^2(i_p) = c\left(\frac{1}{n^p}\right)^2 .$$

Since we are supposing the (n,N) process a regular one, this last equality holds for every $p \in \mathbf{N}$.

Then, in order to construct Ω^p from Ω^{p-1}, the iteration of the (n,N) process adds to the area Ω^{p-1}, an area equal to $c\, 1/n^{2p}$ for __each__ segment s in $\partial\Omega^{p-1}$, that is, an area equal to

$$N^{p-1}\, c\frac{1}{n^{2p}} = \frac{c}{N}\frac{N^p}{n^{2p}} = \frac{c}{N}\frac{(n^2)^p}{n^{2p}} = \frac{c}{N} \quad ,$$

__independent__ of p.

That means: $\mu^2(\Omega^p) \to \infty$ when $p \to \infty$; contradicting the fact: Ω is bounded.

In consequence, associated with any such segment s in $\partial\Omega^{p-1}$ there must exist a region $i_p \subset \Omega^p \cap c\Omega^{p-1}$ (here cA means: the complement of the set A) __and an $e_p \subset \Omega^{p-1} \cap c\Omega^p$__ as well.

For instance, let us consider the simple case $n=2$, $N=n^2=4$, as seen in figure (2)(a).

Then, the sets i_p and e_p are the triangles in $c\Omega^{p-1}$ and Ω^{p-1}, respectively, as shown in figure (2)(b). We observe, in this example, that $\mu^2(i_p) = \mu^2(e_p)$, and this is due to the fact that the shapes of both i_p and e_p are the same.

Now we prove that this is a general situation:

Lemma : If $\partial\Omega$ is produced by a regular $(n,N) = (n,n^2)$ process, then

$$\mu^2(i_p) = \mu^2(e_p) \ .$$

For, suppose that $\mu^2(i_p) > \mu^2(e_p)$. Then, because of the regularity of the process, we would have different constants c_i and c_e such that, for every $p \in \mathbb{N}$ we have:

$$\mu^2(i_p) = c_i \left(\frac{1}{n^p}\right)^2 \ ;$$

$$\mu^2(e_p) = c_e \left(\frac{1}{n^p}\right)^2 \ .$$

Then, $\mu^2(i_p) > \mu^2(e_p)$ implies $c_i > c_e$.

Let $c = c_i - c_e > 0$.

In order to construct Ω^p from Ω^{p-1} we add to the area of Ω^{p-1}, an area equal to $c\ 1/n^{2p}$ for <u>each segment</u> s in $\partial\Omega^{p-1}$; that is, as before, an area $N^{p-1} c\ 1/n^{2p} = c/N$ <u>independent of p</u> ; again as before, that means

$$\mu^2(\Omega^p) \to \infty \text{ when } p \to \infty \ ,$$

but we restricted ourselves to <u>bounded</u> sets throughout.

Now, in our previous example $(n,N) = (n,n^2) = (2,4)$, let us construct the first few Ω^p , for $p = 0 , 1 , 2$, as seen in figure (3).

Notice, in figure (3)(b), that the triangle ABC is entirely contained in Ω^1 , but then, in figure (3)(c), when constructing Ω^2 from Ω^1 , ABC has been subdivided into four triangles, and only

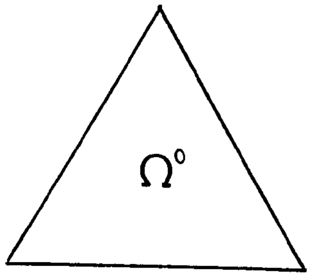


Figure 3(a)

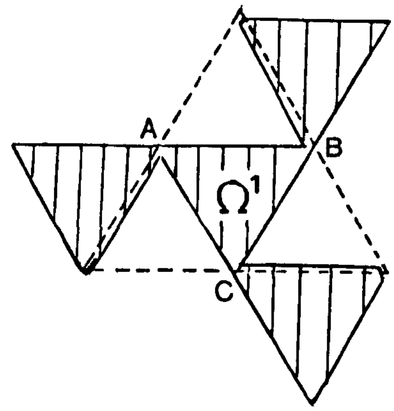


Figure 3(b)

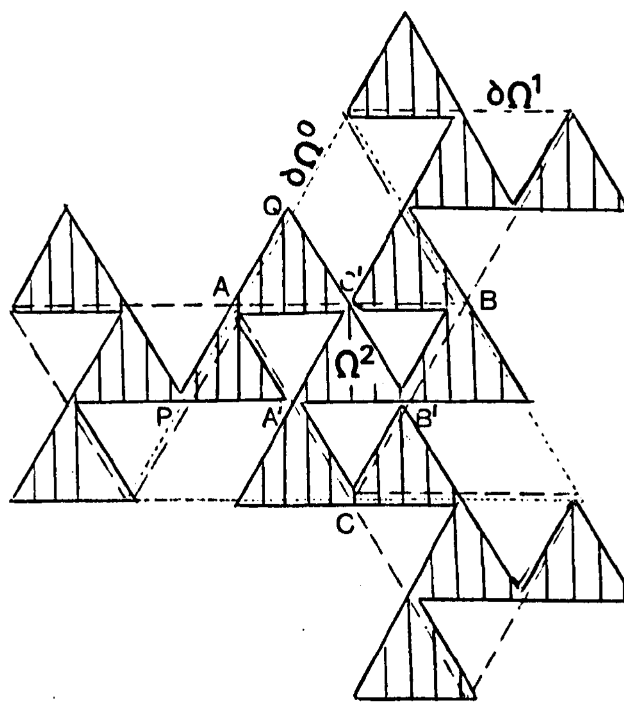


Figure 3(c)

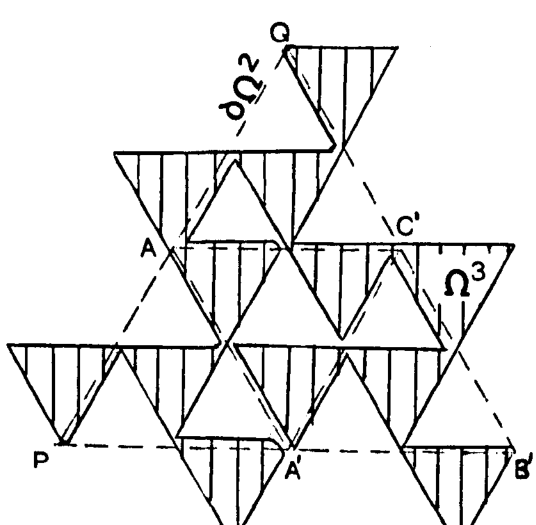


Figure 3(d)

the central one A'B'C' is contained in Ω^2.

Conversely, in figure (3)(c) the triangle AC'A' is entirely contained in the complement of Ω^2; but in figure (3)(d) we see that it has been subdivided into four triangles ... only the central one in the complement of Ω^3.

By the self similarity of the process, this situation is general, so, for a triangle like ABC, we will have, for succesive values of $p \in \mathbf{N}$, the situation described in figure (4), (a), (b), and (c): we can see that

$$\lim_{p \to \infty} \mu^2(\mathrm{Int}\Omega^p \cap \overset{\triangle}{ABC}) = \tfrac{1}{2}\mu^2(\overset{\triangle}{ABC}) \quad ;$$

and also

$$\lim_{p \to \infty} \mu^2(\mathrm{Ext}\Omega^p \cap \overset{\triangle}{ABC}) = \tfrac{1}{2}\mu^2(\overset{\triangle}{ABC})$$

We will prove, next, that this is a general result.

For brevity, let I_p and E_p denote the interior and exterior of Ω^p respectively.

Observation: Notice that, in our example, in which $(n,N) = (n,n^2) = (2,4)$, we have

$$\overset{\circ}{\Omega} = \emptyset \quad ; \quad \Omega = \mathbf{R} \quad .$$

But, if we carry the same (2,4) process of replacement with an initial Ω^o of <u>six</u> sides of unit length as shown in figure (1)(a), then

$$\mu^2(\overset{\circ}{\Omega}) \neq 0$$

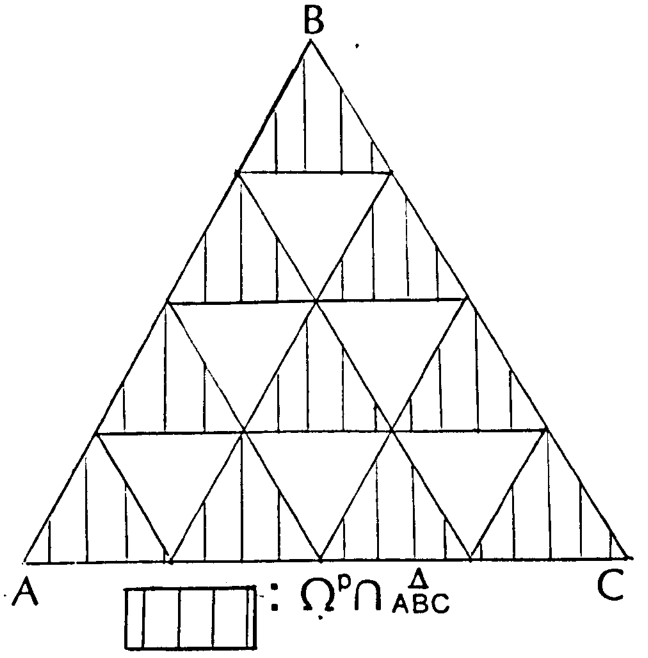


Figure 4(a)

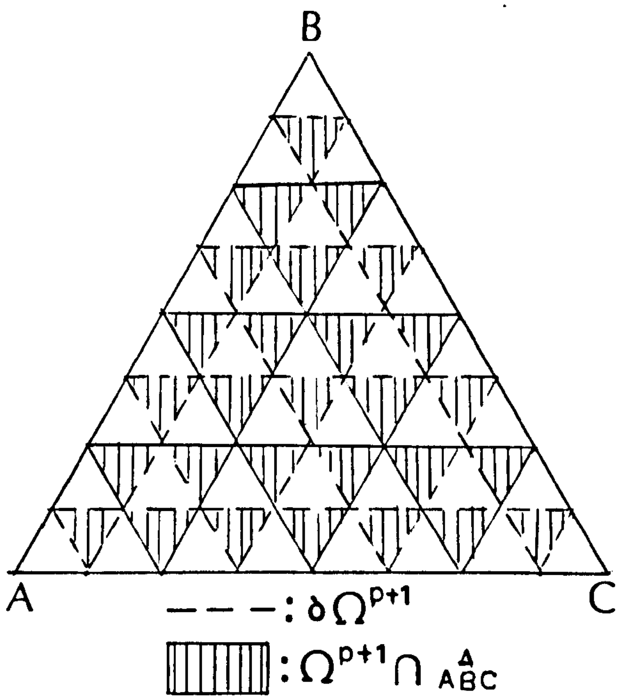


Figure 4(b)

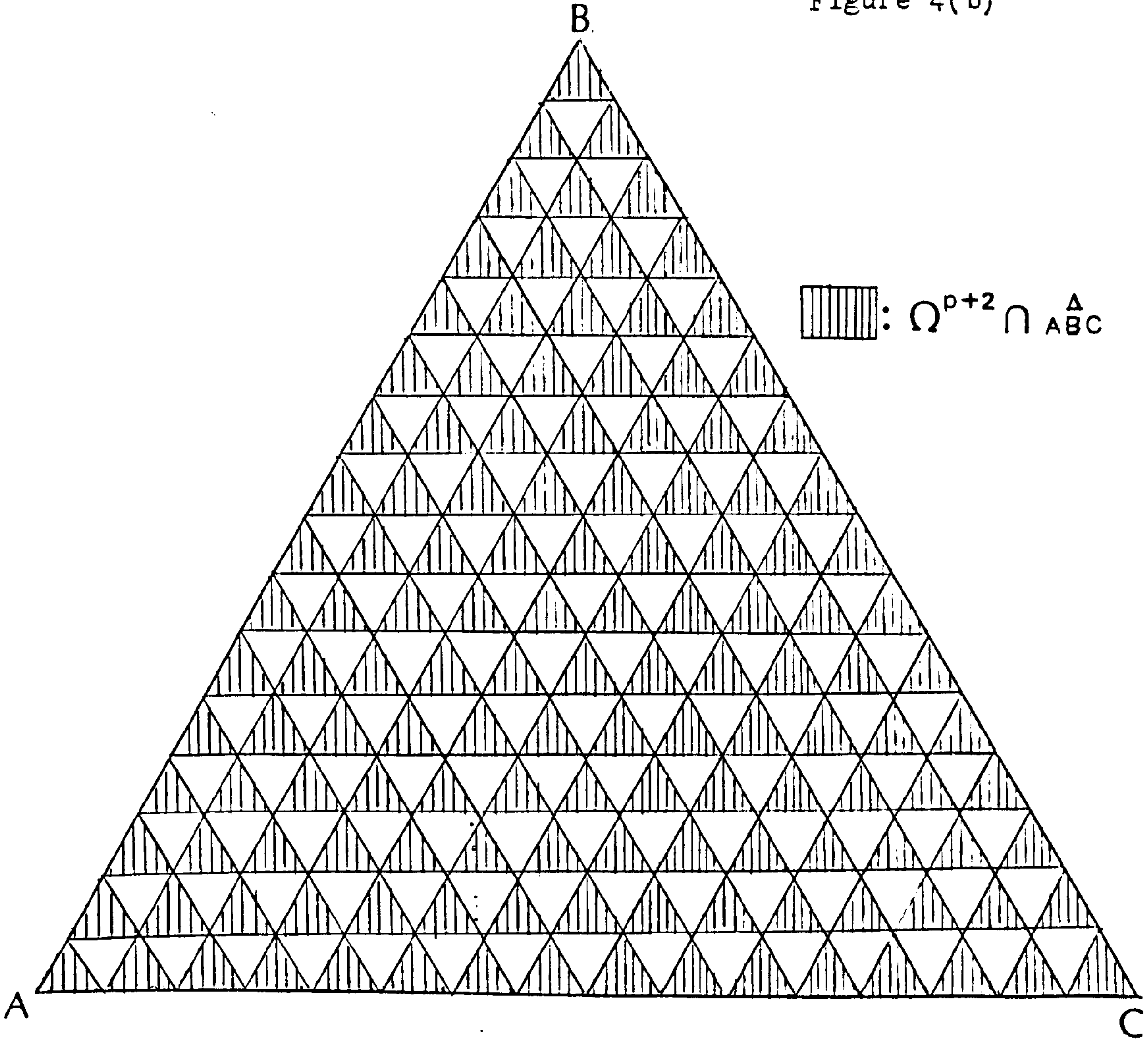


Figure 4(c)

as seen in figure (1)(b); notice, for instance, in this figure, that the triangle ABC fulfills

$$\mu^2(\overset{\triangle}{ABC}) \neq 0 \quad ; \quad \overset{\triangle}{ABC} \subset \mathring{\Omega} \ .$$

Theorem 1

Let $Q = Q(l)$ be a square of side 1. Let $Q \subset R$. Then,

$$\lim_{p \to \infty} \mu^2 [Q(1) \cap I_p] = c(\Omega)\, \mu^2 [Q(1)] \quad .$$

Proof : We begin by proving that :

There exist constants $c_i > 0$, $c_e > 0$, and $p_o \in \mathbb{N}$ such that, for all $p \geq p_o$ we have:

$$\frac{\mu^2 [Q(1) \cap I_p]}{\mu^2 [Q(1)]} \geq c_i \quad ; \quad \frac{\mu^2 [Q(1) \cap E_p]}{\mu^2 [Q(1)]} \geq c_e \quad .$$

It is enough to show the statement for I_p, since the treatment of the cases I_p and E_p is the same.

When forming I_{p+1} from I_p, in the general case, we add, non rampantly, an i_{p+1} for every segment in $\partial\Omega^p$ --and remove an e_{p+1} at the same time.

If $R \subset Q$, then, by our lemma, we have:

$$\mu^2 [\, I_{p+1} \,] = \mu^2 [\, I_p \,] \,, \tag{1}$$

hence the value of $\mu^2 [\, I_p \,]$ is independent of p, and

$$\mu^2 [\, I_{p+k} \cap Q \,] = \mu^2 [\, I_p \cap Q \,] \text{ for every } k \in \mathbb{N} \,.$$

Then we would have, for every p

$$\mu^2 [\, I_p \cap Q \,] \geq \mu^2 [\, I_1 \,] \tag{1'}$$

and then,

$$\frac{\mu^2 [I_p \cap Q]}{l^2} \geq \frac{\mu^2 [I_1]}{l^2} = c_i \quad ,$$

and our claim would be proved.

Now, since we do not have $R \subset Q$, but $Q \subset R$, we do not have the statement (1'), and must find a substitute of it.

We need some notation:

Let M be a positive arbitrary constant; $M >> 1$; let p_0 be so large as to insure

$$\operatorname{diam}(i_{p_0}) \frac{1}{1-n^{-1}} < \frac{1}{2M} \quad ;$$

$$\operatorname{diam}(e_{p_0}) \frac{1}{1-n^{-1}} < \frac{1}{2M} \quad ;$$

where n is the integer in the (n,N) process of replacement.

Let

$$Q' = Q(1-\frac{1}{2M}) = Q(1') \subset Q(1)$$

the square centered at the barycentre of Q , with sides parallel to the ones of Q ; but with side $1' < 1$.

Now, for p_0 as above, let

$$I^{o}_{p_0} = I_{p_0} \cap Q' + \sum_{k \in K} i^{k}_{p_0} \quad ;$$

where $k \in K$ implies: $i_{p_0}{}^{k}$ intersects both Q' and $Q - Q'$. See figure (5) for the case of our previous example.

We need to be remainded that, when defining rampant sums of sets, we said that the sets involved could be taken to be either open, closed, or neither. We will take the $i_{p_0}{}^{k}$ to be closed, and the

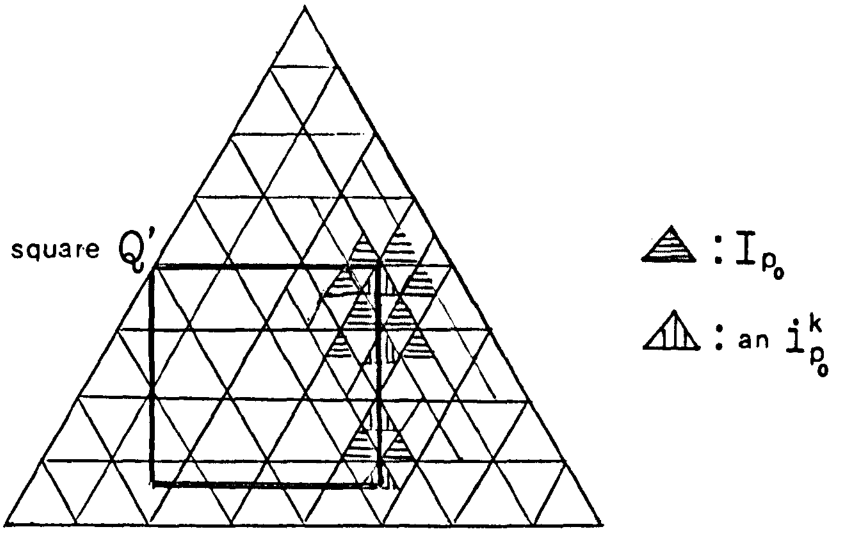


Figure 5

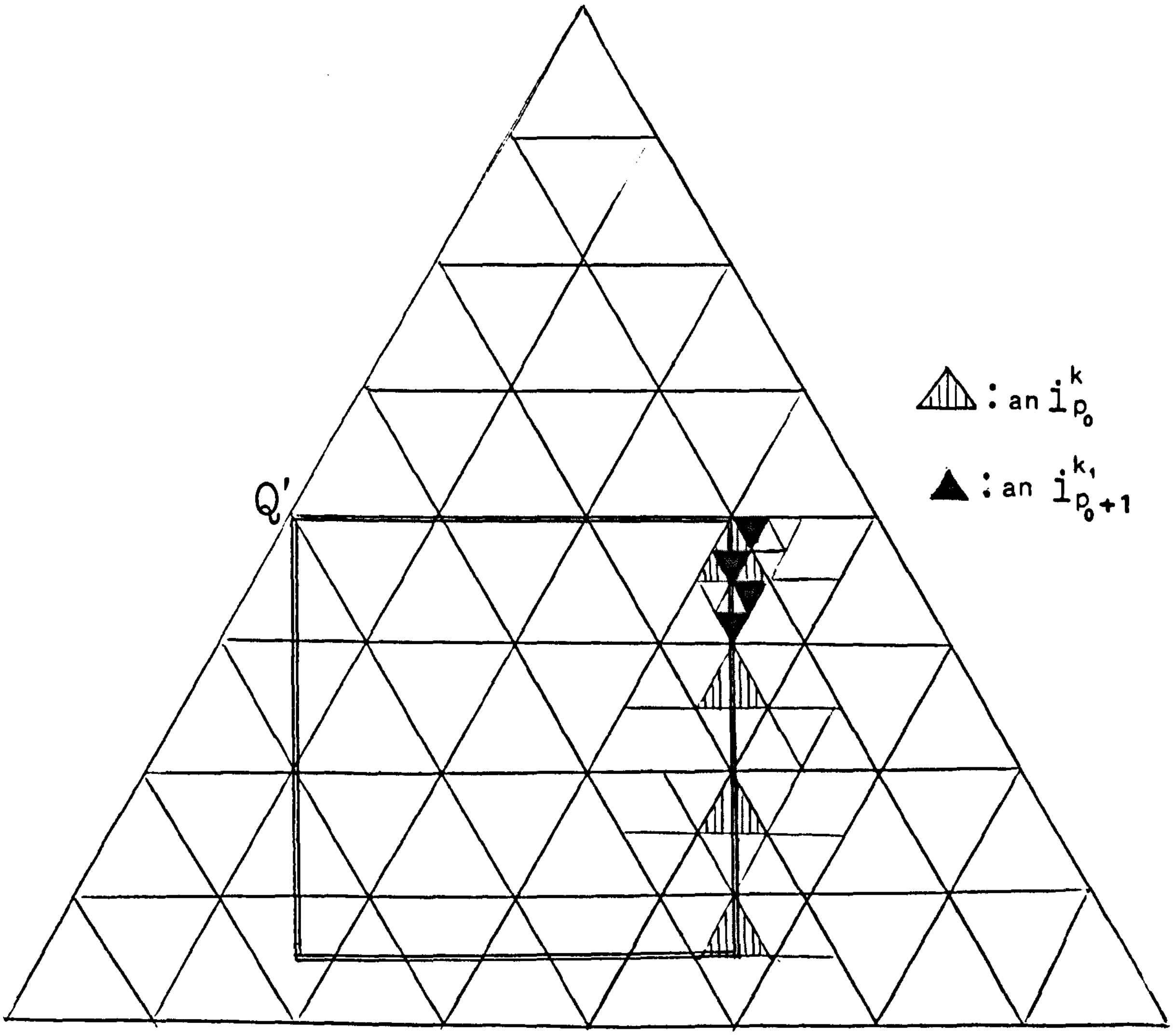


Figure 6

$$i^{k_j}_{p_o+j}$$

to be defined later will be closed as well.

For brevity, we will describe this situation by saying that <u>the</u> $i_{p_o}^{k}$ <u>are adjacent to</u> Q' .

Next, we define:

$$I^{1}_{p_o+1} = I_{p_o+1} \cap Q' + \sum_{k_1 \in K^1} i^{k_1}_{p_o} \quad ,$$

where $k_1 \in K^1$ implies:

$$i^{k_1}_{p_o+1} \text{ intersects some } i^{k_1}_{p_o} \text{ and } Q-Q' \ ;$$

and we will say that the

$$i^{k_1}_{p_o+1} \quad , \ k_1 \in K^1$$

are adjacent to the $i_{p_o}^{k}$, see figure (6) for our example.

We iterate the process defining further

$$I^{k}_{p_o+k} \quad .$$

Observe that, by adding, non rampantly, at the stage k of the construction, all the

$$i^{k}_{p_o+k} \text{ adjacent to } i^{k-1}_{p_o+k-1} \quad ,$$

we ensure:

$$\mu^2 [I^{k}_{p_o+k}] \geq \mu^2 [I_{p_o} \cap Q'] \qquad (2)$$

for every $k \in \mathbb{N}$, which is the statement analogous to (1') for the square Q'.

All the squares that will be dealt with next, will be considered centered at the barycentre of Q, and with sides parallel to the ones of Q.

Notice now that

$$I^{o}_{p_0} \subset Q[l'+\mathrm{diam}(i_{p_0})] \subset Q[l'+\mathrm{diam}(i_{p_0})\frac{1}{1-n^{-1}}] \subset Q(l) = Q \;;$$

also

$$I^{1}_{p_0+1} \subset Q[l'+\mathrm{diam}(i_{p_0})+\frac{\mathrm{diam}(i_{p_0})}{n}] \subset Q[l'+\mathrm{diam}(i_{p_0})\frac{1}{1-n^{-1}}]$$

$$\subset Q(l) = Q \;;$$

and, in general

$$I^{k}_{p_0+k} \subset Q[l'+\mathrm{diam}(i_{p_0})\{1+1/n+1/n^2+\ldots+1/n^k\}]$$

$$\subset Q[l'+\mathrm{diam}(i_{p_0})\frac{1}{1-n^{-1}}] \subseteq Q \;;$$

that is:

$$I^{k}_{p_0+k} \subset Q \quad \text{for every } k \in \mathbb{N} \;;$$

then certainly,

$$I^{k}_{p_0+k} \subset I_{p_0+k} \cap Q \quad \text{for every } k \in \mathbb{N} \qquad (3)$$

Now let us choose any $p \in \mathbb{N}$; $p > p_0$.

For some $k \in \mathbb{N}$ we have: $p = p_0 + k$. Using (3) we have:

$$\mu^2(I_p \cap Q) = \mu^2(I_{p_0+k} \cap Q) \geq \mu^2(I^k_{p_0+k}) \quad ;$$

and using (2) we have:

$$\frac{\mu^2[I_p \cap Q]}{l^2} \geq \frac{\mu^2[I^k_{p_0+k}]}{l^2} \geq \frac{\mu^2[I_{p_0} \cap Q']}{l^2} \quad ;$$

and this constant

$$\frac{\mu^2[I_{p_0} \cap Q']}{l^2}$$

is <u>independent</u> of p, and it is strictly positive (there is certainly at least one such i_{p_0} contained in Q', and $0 \neq \mu^2[i_{p_0}] < \mu^2[I_{p_0} \cap Q']$; in fact Q' contains of the order of N^{p_0} of such sets i_{p_0}).

If we let

$$c_i = \frac{\mu^2[I_{p_0} \cap Q']}{l^2}$$

we have

$$\frac{\mu^2[I_p \cap Q]}{l^2} \geq c_i \quad \text{for every } p > p_0 \quad ,$$

which is our claim.

In order to prove that

$$\lim_{p \to \infty} \frac{\mu^2[I_p \cap Q(1)]}{l^2}$$

exists, we have to prove that the sequence

$$\{A_p\}_{p\in N} \quad ; \quad A_p = \frac{\mu^2[I_p \cap Q(l)]}{l^2}$$

is a Cauchy sequence.

Let $\varepsilon > 0$ be arbitrary, we will find $p_o \in N$ such that

$$|A_p - A_{p_o}| < \varepsilon \quad \text{for every } p \geq p_o \ .$$

We will choose $\varepsilon << \sup\{c_i ; l ; l^2\}$

Let $Q'' = Q(l'') \subset Q = Q(l)$, as before, be a square centered where Q is, its sides parallel to the sides of Q ; and let

$$\frac{\mu^2[Q-Q'']}{l^2} < \varepsilon/3 \ .$$

Let p_o be such that

$$\frac{1}{1-n^{-1}} \sup\{\mathrm{diam}(i_{p_o}) ; \mathrm{diam}(e_{p_o})\} < \frac{l-l''}{2} \ .$$

For any set S such that $Q'' \subset S \subset Q$ we denote

$$B_p = B_p(S) = \frac{\mu^2[I_p \cap S]}{l^2} \ .$$

Then

$$|A_p - B_p(S)| \leq \frac{\mu^2[Q-Q'']}{l^2} < \varepsilon/3 \qquad (1)$$

for every set S such that $Q'' \subset S \subset Q$ and for every $p \geq p_o$.

Now, for each $p > p_o$, we have $p = p_o + k$, for some $k \in$

$\mathbb{N}$, and we will choose S , $Q'' \subset S \subset Q$ as follows:

$$S = I^{k}_{p_0+k}$$

of our claim. Then, again,

$$| B_p - B_{p_0} | < \frac{\mu^2 [Q - Q'']}{l^2} < \varepsilon/3 \quad ,$$

and then, remembering (1), we have:

$$| A_p - A_{p_0} | \le | A_p - B_p | + | B_p - B_{p_0} | + | B_{p_0} - A_{p_0} | < 3 . \varepsilon/3 = \varepsilon \ .$$

Corollary :

Let us assume the hypotheses of theorem 1. Under mild condition imposed on the formation of the Ω^p , $p \in \mathbb{N}$, we can calculate the constant of theorem 1. Namely, let us consider the e_p and the i_p as <u>open</u>, and let us assume that no i_j (or e_l) can intersect <u>both</u> an e_p and an i_p at the same time, for the same p ; and that

$$\mu^1(\partial(I_{p+2} \cap E_p) \cap \partial E_p) = 0 \qquad \text{for every } p \in \mathbb{N} \qquad (1)$$

see figure (7) for an example.

Let us now calculate

$$\frac{\lim\limits_{p \to \infty} \mu^2 [I_p \cap Q(1)]}{\lim\limits_{p \to \infty} \mu^2 [E_p \cap Q(1)]}$$

(we know, by our claim, that both limits are different from zero).

For brevity, let us assume the size of Q normalized so that $l = 1$.

Let ε be sufficiently small, i.e.

$$\varepsilon << \sup\{\lim_{p\to\infty}\mu^2(E_p\cap Q)\,;\,\lim_{p\to\infty}\mu^2(I_p\cap Q)\}\ ;\ \varepsilon<<1=1\ .$$

Let, as before, $Q' = Q\,(l') \subset Q$ be such that

$$\frac{\mu^2[Q-Q']}{l^2} << \varepsilon\ .$$

Then, by estimating

$$\lim_{p\to\infty}\mu^2[E_p\cap Q] \text{ and } \lim_{p\to\infty}\mu^2[I_p\cap Q] \text{ by}$$

$$\lim_{p\to\infty}\mu^2[E_p\cap S] \text{ and } \lim_{p\to\infty}\mu^2[I_p\cap S]$$

for any set S such that $Q'\subset S\subset Q$, we would be working with an arbitrarily good approximation.

Let p be, as before, such that

$$\sup\{\mathrm{diam}(i_p)\,;\,\mathrm{diam}(e_p)\} << \frac{l-l'}{2}\quad ,$$

and, for brevity, assume that E_p and I_p denote $E_p\cap Q$ and $I_p\cap Q$ in the remainder of this proof.

Then, with that arbitrarily good approximation we have:

$$E_p = I_{p+1}\cap E_p + I_{p+2}\cap E_{p+1}\cap E_p + \dots$$

Let us consider, first, $I_{p+1}\cap E_p$. For that purposse, we consider each $i_{p+1}\cap E_p$, $N^{\#}$ being the number of segments in ∂I_p . Then we have $N^{\#}$ of such $i_{p+1}\cap E_p$ and therefore

$$\mu^2[I_{p+1}\cap E_p] = N^{\#}\,\mu^2[i_{p+1}]\ .$$

Let N^i be the number of segments in $\partial i_{p+1} \cap E_p$ which are <u>contained in</u> <u>Int E_p</u> ; and consider next $I_{p+2} \cap E_{p+1} \cap E_p$ by focussing again on the different i_{p+2} contained in $E_{p+1} \cap E_p$.

The sets i_{p+2} associated with each of those N^i segments (in the boundary of i_{p+1} included in Int E_p) will be contained in E_p (and, of course, in E_{p+1}); and we will have $N^i N^{\#}$ of those; and the other sets i_{p+2} <u>not</u> associated with those N^i segments, will be contained in I_p , and, therefore, we are <u>not</u> interested in them.

Therefore,

$$\mu^2 [I_{p+2} \cap E_{p+1} \cap E_p] = N^i N^{\#} \mu^2 [i_{p+2}] = \frac{N^i N^{\#}}{n^2} \mu^2 [i_{p+1}] \ .$$

Next, let us consider

$$I_{p+3} \cap E_{p+2} \cap E_{p+1} \cap E_p \quad ,$$

also by focussing in different i_{p+3} stemming from segments in the boundary of the i_{p+2} considered above.

Now, for such an i_{p+2} , we know that

$$\mu^1 [\partial(i_{p+2}) \cap \partial E_p] = 0 \quad ,$$

by condition (1), and therefore that i_{p+2} is entirely contained in Int e_p <u>together with its boundary</u> , (except perhaps for some end points of a segment in the boundary); see figure (7) for our example.

Also, by the regularity of the fractal process, N^i segments of its boundary are also in Int E_{p+1} .

Therefore, these N^i sets i_{p+3} associated with this particular

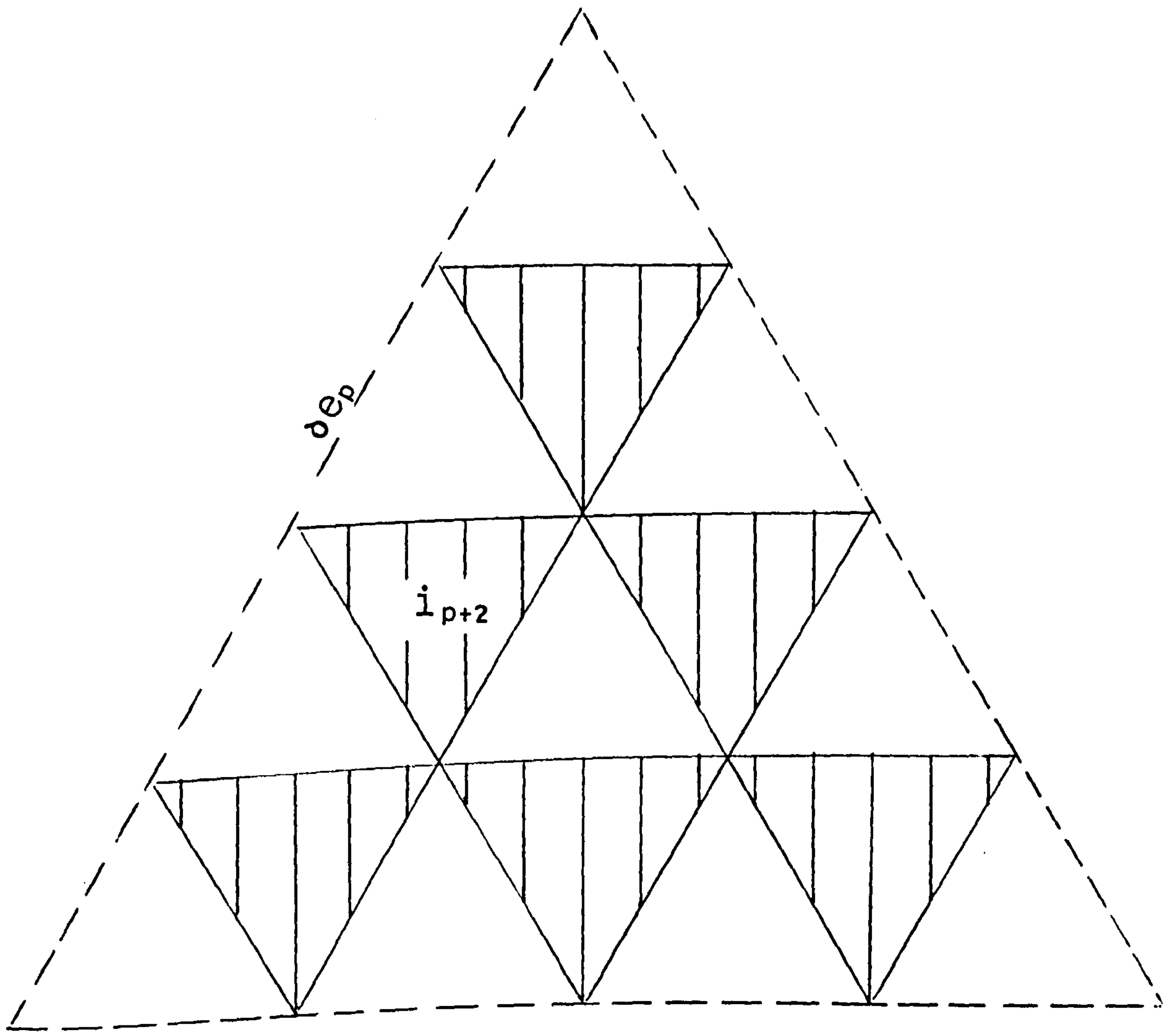


Figure 7

i_{p+2} are in Int E_p ; in Int E_{p+1} ; and, of course, in E_{p+2} ; and we have exactly N^i of them for each of the $N^i N^\#$ sets i_{p+2} considered, i.e. we have $N^\#(N^i)^2$ of sets i_{p+3} considered.

Therefore

$$\mu^2 [I_{p+3} \cap E_{p+2} \cap E_{p+1} \cap E_p] = N^\# [N^i]^2 \mu^2 [i_{p+3}] =$$

$$\frac{N^\# N^{2i}}{(n^2)^2} \mu^2 [i_{p+1}] \quad .$$

The regularity of the fractal process allows us to iterate this result to obtain, finally:

$$\mu^2 [E_p] = \sum_{k=0}^{\infty} \frac{c N^\# [N^i]^k}{[n^{p+1}]^2 [n^2]^k} = \frac{c N^\#}{[n^{p+1}]^2} \frac{1}{1 - N^i/n^2} \quad .$$

In a similar manner

$$\mu^2 [I_p \cap Q] = \frac{c N^\#}{[n^{p+1}]^2} \frac{1}{1 - N^e/n^2} \quad .$$

where N^e is similarly defined.

Therefore

$$\frac{\mu^2 [I_p \cap Q]}{\mu^2 [E_p \cap Q]} = \frac{1 - N^e/n^2}{1 - N^i/n^2} \quad ,$$

hence, the corollary follows.

Now we can prove:

Theorem 2 :

Let U_Ω be a neighborhood of the clausure of Ω . Let $f \in L^1(U_\Omega)$. Then, there is a constant $c \in (0,.1)$ such that

$$\lim_{p\to\infty} \iint_{\Omega^p} f = \iint_{\mathring{\Omega}} f + c \iint_{\mathbf{R}} f .$$

Proof :

There exists $p' \in \mathbb{N}$ such that

$$p > p' \text{ implies } \Omega^p \subset U_\Omega .$$

For the remainder of the proof we will restrict ourselves to $p > p'$.

Let us now suppose that there exists $c \in (0, 1)$ such that

$$\lim_{p\to\infty} \iint_{\Omega^p} f^s = \iint_{\mathring{\Omega}} f^s + c \iint_{\mathbf{R}} f^s \qquad (1)$$

for every simple function $f^s \in L^1(U_\Omega)$. Then, the theorem is true for every function $f \in L^1(U_\Omega)$, with the same constant c of (1): for, let $\varepsilon > 0$ be arbitrary, and, given $f \in L^1(U_\Omega)$, choose a simple function f^s , such that

$$\| f^s - f \|_{L^1(U_\Omega)} < \varepsilon/3 .$$

Then

$$\left| \lim_{p\to\infty} \iint_{\Omega^p} f^s - \lim_{p\to\infty} \iint_{\Omega^p} f \right| \leq \varepsilon$$

It remains to prove the theorem for simple functions. But, since a simple function is but a linear combination of characteristic functions of squares, then it is enough to prove the theorem for characteristic function of squares.

Let, then χ_Q be such a function, Q is a square. By subdividing, if necessary, the square Q in smaller ones Q_i we can ensure: either

$$Q_i \subset \mathbf{R} \ , \ \text{or} \ Q_i \subset \mathring{\Omega} \ , \ \text{or} \ Q_i \subset \mathbb{R}^2 - [\mathbf{R} + \mathring{\Omega}] = \mathbb{R}^2 - \Omega = E \ ;$$

and let us denote the corresponding indeces i , j , k , and let us group them as belonging to the corresponding sets of indeces

$$I_{\mathbf{R}} \ ; \ I_{\mathring{\Omega}} \ ; \ \text{and} \ I_E \ .$$

Hence,

$$\chi_Q = \sum_{i \in I_{\mathbf{R}}} \chi_{Q_i} + \sum_{j \in I_{\mathring{\Omega}}} \chi_{Q_j} + \sum_{k \in I_E} \chi_{Q_k} \ .$$

Then,

$$\lim_{p \to \infty} \iint_{\Omega^p} \chi_Q = \sum_{i \in I_{\mathbf{R}}} \lim_{p \to \infty} \mu^2 [\Omega^p \cap Q_i] +$$

$$\sum_{j \in I_{\mathring{\Omega}}} \lim_{p \to \infty} \mu^2 [\Omega^p \cap Q_j] + \sum_{k \in I_E} \lim_{p \to \infty} \mu^2 [\Omega^p \cap Q_k] \ .$$

Now, when $p \to \infty$ we have:

$Q_i \subset \mathbf{R}$ implies that we can apply Theorem 1, and

$$\lim_{p \to \infty} \mu^2 [\Omega^p \cap Q_i] = c\,\mu^2(Q_i) \ ;$$

$$Q_j \subset \mathring{\Omega} \quad \text{implies}$$

that for p big enough (and with a convenient subdivision of Q) we will have: $Q_j \subset \Omega^p$, and then

$$\mu^2 [\Omega^p \cap Q_j] = \mu^2 (Q_j) \quad ;$$

and similarly, $Q_k \subset E$ implies that, for p big enough we will have

$$Q_k \cap \Omega^p = \emptyset \quad , \text{ therefore}$$

$$\mu^2 [\Omega^p \cap Q_k] = 0 \quad .$$

Then,

$$\lim_{p \to \infty} \iint_{\Omega^p} \chi_Q = c \sum_{i \in I_{\mathbf{R}}} \mu^2 [Q_i] + \sum_{j \in I_{\mathring{\Omega}}} \mu^2 [Q_j] =$$

$$c \iint_{\mathbf{R}} \sum_{i \in I_{\mathbf{R}}} \chi_{Q_i} + \iint_{\mathring{\Omega}} \sum_{j \in I_{\mathring{\Omega}}} \chi_{Q_j} =$$

$$c \iint_{\mathbf{R}} \chi_Q + \iint_{\mathring{\Omega}} \chi_Q \quad ;$$

since we have:

$$\chi_{Q_j} = 0 \quad \text{in} \quad \mathbf{R} \; ;$$
$$\chi_{Q_i} = 0 \quad \text{in} \quad \mathring{\Omega} \; .$$

Finally, notice that, when the dimension $d \in (1, 2)$, we can write:

$$\iint_{\Omega^\varepsilon} f = \lim_{p\to\infty} \iint_{[\Omega^p]^\varepsilon} f \quad ; \quad \iint_{\Omega} f = \lim_{p\to\infty} \iint_{\Omega^p} f \quad .$$

Let us <u>define</u>, then, for our case $d = 2$,

$$\iint_{\Omega^\varepsilon} f = \lim_{p\to\infty} \iint_{[\Omega^p]^\varepsilon} f \quad ; \quad \iint_{\Omega} f = \lim_{p\to\infty} \iint_{\Omega^p} f \quad ;$$

and let us now calculate our quotient $Q^\varepsilon(f)$ of the Rellich theorem for, say, $f \equiv 1$. We obtain

$$Q^\varepsilon(f) = \frac{\lim\limits_{p\to\infty} \mu^2[(\Omega^p)^\varepsilon]}{\lim\limits_{p\to\infty} \mu^2(\Omega^p)} \quad .$$

If we consider p so big as to ensure $\operatorname{diam}(i^p) < \varepsilon$ and $\operatorname{diam}(e^p) < \varepsilon$ we have:

$$\mu^2[(\Omega^p)^\varepsilon] = \mu^2\{(\Omega^p - \mathring{\Omega}) + (\mathring{\Omega})^\varepsilon\} = \mu^2(\Omega^p - \mathring{\Omega}) + \mu^2[(\mathring{\Omega})^\varepsilon]$$

which tends to $c\,\mu^2(R) + c'\,\varepsilon^{2-d}$, using our theorems, and d being here the dimension of

$$\partial\mathring{\Omega} \; .$$

Therefore

$$\Gamma^\varepsilon \sim \frac{c\,\mu^2(R)}{c\,\mu^2(R) + \mu^2(\mathring{\Omega})} \quad :$$

notice that this result is given by our former value of $c\,\varepsilon^{2-d}$ when $d = 2$ ...

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Epilogue

Classical results on P.D.E. acting on functions defined on a bounded region R, need the smoothness of the boundary ∂R, not only because they need to apply the Rellich theorem (of course), but also, for instance, the Sobolev extension theorem; classical texts on P.D.E., like Agmon's, actually state both theorems at the beginning.

As commented in the Introduction, hard analysis may use the smoothness of ∂R also at many a stage of any proof, while soft analysis tends to work in an abstract space of operators, needing the smoothness of ∂R only in order to apply concrete theorems--like Rellich's or Sobolev's. An example of this last type of work is Beals's [1]. In fact, Beals [1] needs smoothness on the boundary of the regions he works with, because, in that paper, he uses both the classical Rellich theorem and the theorem of extension of Sobolev.

The theorem of extension of Sobolev states that, if the boundary of a bounded region R is very smooth, and if $R \subset S$, S a square, then we can extend a function f supported in R to a function f' supported in S. The function f' has the same degree of L^p_k-smoothness as f; and the norms of both functions are comparable.

In this thesis we have extended the theorem of Rellich to include regions with (regular) fractal boundaries, ...but we will probably not be able to extend the theorem of extension of Sobolev to include fractal boundaries, since it is easy to find smooth functions f supported in the open region Ω , $\partial\Omega$ a regular fractal, that will not even admit a smooth extension f' supported in cl.Ω .

When we examine the work we have been quoting, we see that the theorem of extension is used in the following way: an orthonormal basis for $L^2{}_2(R)$, for $L^2{}_1(R)$ and for $L^2(R)$ is needed, with certain specific properties. Now, if R were a square, the standard basis of sines and cosines would certainly have those needed properties. So the argument goes as follows: the region R (with smooth boundary) is placed inside a square S , and the functions in $L^2{}_2(R)$, in $L^2{}_1(R)$ and in $L^2(R)$ are extended to functions in S , with the same degree of smoothness and comparable norm..., so the standard basis for the square S will now work for R .

In short, we assimilate R to the square S in a standard way in which we cannot possibly assimilate Ω , $\partial\Omega$ a fractal. This is a first problem.

Now, the work quoted above is concerned with an estimation of the distribution of the eigenvalues of an elliptic operator acting on functions defined on R. This means that the standard argument just described reduces the study of the general case of R to the case of a square.

Notice, though, that the distribution of the eigenvalues just referred to, changes dramatically with an alteration of the shape of ∂R ... so, in a way, with that standard argument, we are assimilating the study of <u>very</u> different (among themselves) cases to the case of the square. This is a second problem.

A way out of both problems has been found to be the following: To forget about that square and to construct a basis (with those convenient properties) for $L^2{}_2(R)$, $L^2{}_1(\Omega)$ and $L^2(\Omega)$, $\partial\Omega$ a fractal;... and, in general: To devise a technique to construct bases for $L^2{}_2(R)$, $L^2{}_1(R)$ and $L^2(R)$ (for regions R for which ∂R is smooth) which will have different properties for different shapes of different ∂R --that is, that will discriminate among different shapes of ∂R .

The technique is as follows:
Let us take, as an example, the Ω depicted in page 51: a Minkowski sausage of dimension $d = \log 5/\log 3$. Such an Ω can be seen as a non rampant sum of infinite squares s_i of decreasing size.

In each square $s = s_i$ we construct the orthonormal basis for $L^2{}_2(s)$, $L^2{}_1(s)$ and $L^2(s)$ which is based on the two functions $\sin(nx\pi/a)$ and $\cos((n+\frac{1}{2})x\pi/a)$, where a is the side of the square s we are working on.

Notice that the functions in this basis are zero in the boundary of s . Therefore, we can extend these functions to the rest of Ω with the value zero, and we have them now defined in all of Ω .

Repeating the same procedure with any other square s_j , and collecting all those functions defined on Ω (and associated with each of the original squares s_j), we obtain a complete orthonormal basis for $L^2(\Omega)$ and $L^2{}_1(\Omega)$. Let us observe, though, that these functions are not in $L^2{}_2(\Omega)$: they do not have $L^2{}_2(\Omega)$-smoothness at the boundary of the square s in which they originated.

The next step is, therefore, to take the functions originated in each square s, and modified them (inside s) near the boundary of s, so that the modified functions tend to zero (when approaching ∂s) with the necessary degree of smoothness.

If the modification of the functions is slight enough, the new set of functions is still a complete basis for all those Sobolev spaces.

But that basis is not orthonormal any more, for the scalar product of two such modified functions (f_i, f_j) based in the same square will not be zero...But if the latter is small enough, i.e. if $(f_i, f_j)=\varepsilon_{i,j}$ is small enough for every i and every j, then we can introduce the concept of a "quasi-orthonormal" basis, and prove that it has, essentially, the properties that we need--e.g. ,the properties (of the standard basis) that were needed in [1].

By having now the $\varepsilon_{i,j}$ not equal to zero, we have introduced an encumbrance: it takes longer to prove results working with this new basis. But we gain an advantage: its properties allow us to discriminate between different shapes of ∂R (∂R smooth), and --with another 190 pages which we thought it better not to include with this thesis--we can also prove and extend new results for regions Ω, $\partial\Omega$ a fractal; ... e.g. the results in [1], and a certain dependence of the results in [1] on the dimension of $\partial\Omega$.